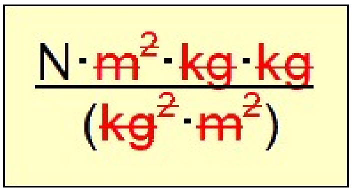
Checking the Units of Engineering Calculations ____________



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1. Scope. This paper describes how to check the units of equations by cancellation.

2. Introduction.

The checking of the units of an equation is a way of ensuring that you have used a consistent set of units in the equation. This is an essential requirement for engineering applications. The failure to use a consistent set of units could at least be embarrassing or at worst dangerous - catastrophic failure of an engineering structure could be costly financially or even get somebody killed.

Dimensional analysis by checking the units of equations by cancellation was something I was taught to do when studying for my Higher National Certificate in Mechanical Engineering at Stow College, Glasgow in the 1960's. In these days the British Imperial system of units was still being used. However, I served my engineering apprenticeship with the British Polar Engines company, who built diesel engines under licence from a Swedish company (NOHAB), and the metric system was in use. The introduction of the International System of Units (SI) reference 1, in 1960, has made the whole question of units much more rational and avoids many of the disadvantages of the Imperial system.

The following procedure and examples explain how to go about this checking of units including some pitfalls to watch out for. I must say, however, that the checking of units is something that is probably easier to teach than to explain on paper. The examples chosen are just a selection from a long career in mechanical engineering.

3. What Units?

First of all let me say that when I talk about units I mean units it the widest possible sense. For example it includes:

- SI Base units such as: length metre m, mass kilogram kg, time second s, thermodynamic temperature kelvin K, etc.
- SI Constants such as: Boltzmann constant J/K, etc.
- Derived units in terms of the base units, such as: velocity m/s, etc.
- Derived units with special names such as: force (weight) in newton N, energy (work) in joule, plane angle in radian, etc.
- Units with special names decimally related to SI units such as: pressure in bar, etc.
- Units with special names exactly defined but not decimally related to SI units such as: angle in degree, etc.
- Non-SI units that are accepted for use with SI units, such as: litre $10^{-3}m^3$, etc.
- Decimal multiples such as: millimetres mm, etc.
- Finally it is possible that the units may cancel out to give a dimensionless number (pure number) such as: Reynolds Number *Re*, etc.

4. Procedure.

The basic procedure necessary to carry out the checking of units by cancellation is as follows:

1. It is best to work with a rational system of units such as the metric SI units.

2. Identify the units on the Left Hand Side (LHS) of the equation.

3. Replace each variable or constant (i.e. factor, coefficient, parameter, or number) on the Right Hand Side (RHS) of the equation with its units.

4. Leave a space or put a half-high dot (centre dot or middle dot) between adjacent units to signify multiplication of units; this is to avoid confusion with decimal multiples. e.g. ms is millisecond but m s or m·s is metre second.

[**Note**: Where necessary, I have used an asterisk ∗ to represent multiplication in equations and when calculating numerical values. I use a slash / or a negative exponent (index or power) to signify division of units, e.g. velocity m/s or $m{\cdot}s^{-1}$. I have used the slash / to represent division in equations and when calculating numerical values. In scientific notation I have used × or E, e.g. 4.905 × 10^7 = 4.905E+7]

5. If any variable or constant on the RHS of the equation is dimensionless and has no units then it can be eliminated (i.e. ignored) from the units check or replace it with a 1 if convenient to do so. But note, as we shall see, there are some variables or constants that are dimensionless but do have units (e.g. plane angle in radians); it can be useful to keep these units in the units check. Any index or indices (exponents or powers), like squares, cubes, or square roots, etc., are generally pure numbers and dimensionless but these need to be kept in the units check otherwise the units will not cancel correctly. It is essential to understand the rules of these indices. A few examples are given in the attached **Appendix 1: Rules of Indices**.

6. Cancel the units on the RHS of the equation by strikethrough those units on the numerator with identical units on the denominator taking care to include any indices. The resulting RHS units should be identical with the LHS units. If they are not identical then something is wrong.

The most likely reasons for something being wrong with the units are:

- Using inconsistent units.
- Wrong use of conversion factors.
- Misinterpretation of the meaning of a symbol.
- Missing index (exponent or power).
- Wrong rearrangement of an equation.
- Using too many decimal multiples resulting in confusing or compound prefixes.
- Typing error in the equation.

The procedure, and how to avoid some of the pitfalls, is best understood by a few examples. These are from mechanical engineering but the procedure should be applicable to other engineering applications.

5. Things You Need to Know.

Before we start with some examples, there are several essential points that need to be highlighted.

1. The various technical references tend to use different notation (symbols) for the variables and constants. Also the sign directions, of the chosen co-ordinate axis, are often different. The result is that each reference, and this text is no different (refer to the attached **Appendix 2: Notation**), has to be studied carefully in order to fully understand the notation in use – get into a habit of doing this. Even within the same document there can be confusion where the same symbol is used in a different context (there plenty of examples in this text). To take a 'simple' example; the symbol g is the unit of mass, the gram, in the SI system with base unit the kilogram kg. However, g is often used as the symbol for the acceleration due to gravity, i.e. $g \approx 9.81$ m/s^2 on planet Earth at sea level. But that is not all; there is common usage to quote pressure in bars, e.g. a pressure vessel designed for a pressure = say 10 bar(g). In this context the (g) stands for gauge pressure, i.e. above atmospheric pressure (on Earth at sea level we are at a gauge pressure of 0 bar) as opposed to 10 bar(a) where the (a) stands for absolute pressure, i.e. gauge pressure + atmospheric pressure (on Earth at sea level we are at an absolute pressure of one atmosphere, which is an absolute pressure of 1.01325 bar or about 14.7 lbf/in^2 in the Imperial system).

> **The lesson here is**: It is necessary to keep your wits about you especially when using unfamiliar notation. Do not assume the notation you are familiar with has the same context as that which you are unfamiliar with. When quoting pressures it is best to spell it out, e.g. at a gauge pressure of 10 bar or 11 bar absolute pressure.

2. The British (the Brits and I am one of them) have a bad habit of abbreviating units. One of the most convenient units in mechanical engineering is the millimetre mm. It is often used as the units of length on engineering drawings. The British have got into the habit of calling the millimetre a mil or mils. This would not be too bad if this was the only context. However, apparently the Americans use the mil as a unit of thickness meaning a thousandth of an inch, i.e. mil = 0.001 inch = 0.0254 mm. A recipe for confusion when you have a meeting between parties from across the Atlantic.

> **The lesson here is**: do not assume that because two parties speak the same language they understand the notation you are familiar with has the same context as that which they are familiar with.

3. Equations as a series of terms often come up in engineering. In many series the terms are dimensionless, e.g. $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$ Here the argument x is the angle in radians and is dimensionless as are the indexes and factorials, $\cos x$ is dimensionless. However, even

the most simple of equations may have units, such as the straight-line equation: $y = a*x + b$. The three groups of terms y, $a*x$ and b all must have identical units. There is no way that the groups of terms on the RHS $a*x + b$ can be added algebraically to give the correct answer unless the groups of terms $a*x$ and b have identical units. If you know the units of the LHS then that sets the units for the terms on the RHS. If you know the units of just one of the groups of terms on the RHS that sets the units for the other groups of terms on the RHS and the units of the LHS.

The lesson here is: In a series of terms always check that the groups of terms of the RHS have identical units and these units are identical to those of the LHS. If the units are not identical then something is wrong.

4. In an ideal world a set of equations would be such that any consistent system of units could be applied and a sensible (correct) result obtained. However, in the real world there are some equations that only give a correct result when specific units are input. Here is an example: consider the following simple equation $b = 2.52\sqrt{b_o}$ which is used in the design of bolted flanges for pressure purposes. The variables b and b_o are gasket seating widths that are smaller than the actual seating width due to an allowance for flange rotation.

   Where: b_o is the basic gasket seating width
   b is the effective gasket seating width
   The number 2.52 is an constant that would appear to have units of √Length in order that the term b has units of Length.

   Working out a couple of results; first in millimetres then in inches, to see if the results are in agreement.

   First input in millimetres. Try b_o = 8.0 mm. Therefore $b = 2.52\sqrt{8.0} = 7.13$ mm (= 0.28 inches). OK seems reasonable.

   Now input in inches. b_o = 0.315" (= 8.0 mm). Therefore $b = 2.52\sqrt{0.315} = 1.414$" (= 35.9 mm). This result does NOT look reasonable and the results are NOT in agreement, clearly something is badly (fatally) wrong.

   What is wrong is that the equation $b = 2.52\sqrt{b_o}$ <u>is only valid when used with dimensions input in millimetres</u> (it will not even work with metres or centimetres – only millimetres). The equation is empirical and has no direct relationship with units.

   If you want to use inches the correct equation to use is: $b = 0.5\sqrt{b_o}$. Therefore, for b_o = 0.315"; $b = 0.5\sqrt{0.315} = 0.28$" (= 7.13 mm). Correct! <u>But this equation only works with dimensions input in inches</u>.

The lesson here is: never mix systems of units into equations unless you are absolutely sure that the equations can be used with any consistent system of units.

5. The checking of units cannot be relied on in every case for establishing if an equation is correct or not. Consider the equation for kinetic energy $E_k = M*V^2/2$. The units of energy on the LHS is the newton metre, N·m. On the RHS the mass M has units of kilogram, kg and velocity V has units of metres per second, m/s. The index 2 and the constant 2 in the denominator are dimensionless pure numbers. The units on the RHS are $kg \cdot m^2/s^2$ which can easily be shown to be a N·m (refer to **Example 6.1** below for an example in the use of base units). There is no great difficulty in showing that these units cancel correctly. However, let us look at two cases where say a typing error might occur:

   Case (a), a missing index. In this case the equation would be $E_k = M*V/2$. A units check would immediately show that something was wrong, as the RHS units would be kg·m/s, which is NOT correct for energy.

   Case (b), a missing constant 2 in the denominator. In this case the equation would be $E_k = M*V^2$. A units check would NOT show that something was wrong, because the missing 2 was a dimensionless constant and would NOT show up on a units check, the RHS units would cancel correctly to give N·m for energy even though the equation was wrong!

The lesson here is: Units checking alone cannot show that an equation is correct and of course units checking will not be able to establish if an equation has been used out of context. If the wrong method of analysis is applied, no amount of units checking will get you the correct answer!

6. Examples.

Example 6.1 Force due to Acceleration.

This is a very simple example that illustrates that there are cases where the units do not initially seem to cancel to anything recognisable.

Suppose we have a 70 kg mass and want to find the weight of this mass on the surface of the Earth at sea level.

We know weight can be calculated as an inertia force using Newton's second law of motion.

Where: force (or weight) $F = M\text{(mass)} * a\text{(acceleration)}$ **eq(1)**

The SI units of force on the LHS of **eq(1)** is the newton with symbol N. The units of mass times acceleration on the RHS of **eq(1)** must cancel to newtons.

Substituting the acceleration due to gravity, $g \approx 9.81\ \text{m/s}^2$.

We have: $F = M * g = 70 * 9.81 = 686.7$ but what are the units of the 686.7 ? We can do a units check and find out.

Check the units. Substitute the units of each term on the RHS of the equation **(1)** and we have: $\text{kg}\cdot\text{m/s}^2$ (or $\text{kg}\cdot\text{m}\cdot\text{s}^{-2}$).

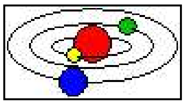
At first glance these units do not cancel to give newtons, so what is a $\text{kg}\cdot\text{m/s}^2$?

Reference to the SI system of units, reference 1, gives the newton in terms of the base units as N = $\text{kg}\cdot\text{m/s}^2$ (or $\text{kg}\cdot\text{m}\cdot\text{s}^{-2}$). Therefore the units on the RHS of equation **(1)** are correct and the weight of a 70 kg mass on the Earth is 686.7 N.

The lesson here is: Sometimes it is convenient to use SI base units. In other cases a check to see if a group of SI base units can be expressed in the form of a single SI derived unit as this can often make more sense from an engineering point of view.

[**Note**: A newton is about 0.225 lbf in the Imperial system so 686.7 N converts to about 154.5 lbf. This is about what I used to weigh with a boiler suit on – well what else would a 1960's engineering apprentice be wearing!]

Example 6.2 Force due to Gravitational Mass.



Let us repeat the above example but this time calculate the force using Newton's law of universal gravitation. We already know the answer from **Example 6.1** but let us proceed anyway; this example will show the basic concept of cancelling the units.

From Newton's law of universal gravitation. The force of attraction F between two masses, m_1 and m_2 at a distance R apart, is given by:

$$F = G * m_1 * m_2 / R^2 \qquad \textbf{eq(2)}$$

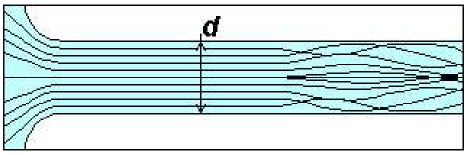
Where: G = the gravitational constant $\approx 6.674\times10^{-11}\ \text{m}^3/(\text{kg}\cdot\text{s}^2) \approx 6.674\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$
m_1= 70 kg mass
m_2= the mass of the Earth $\approx 5.972\times10^{24}$ kg
R = distance between the masses taken as the mean radius of the Earth $\approx 6.371 \times 10^6$ m

[**Note**: Any slight difference in the radius of the Earth and the distance to the centre of the 70 kg mass is ignored. The values of the: gravitational constant, mass of the Earth, mean radius of the Earth and the acceleration due to gravity are all approximations.]

Check the units. The units on the RHS of **eq(2)** are: $\text{N}\cdot\text{m}^2\cdot\text{kg}\cdot\text{kg}/(\text{kg}^2\cdot\text{m}^2) = \text{N}\cdot\cancel{\text{m}^2\cdot\text{kg}\cdot\text{kg}}/(\cancel{\text{kg}^2\cdot\text{m}^2})$. By inspection we see that the units do cancel (strikethrough) correctly to leave N the unit of force in newtons.

Calculate the result. $F = G * m_1 * m_2 / R^2 = 6.674\times10^{-11} * 70 * 5.972\times10^{24}/(6.371\times10^6)^2 = 687.4$ N. This result is very close to **Example 6.1**, as expected.

Example 6.3 Reynolds Number a fluid mechanics problem.



Any serious work on fluid mechanics is likely to involve the application of dimensionless numbers. The most famous being the Reynolds Number. The following example illustrates the use of dimensionless numbers (pure numbers) and highlights some of the pitfalls where confusion with decimal prefixes might be made.

Consider the flow of water in a smooth bore pipe of 50 mm diameter and 100 m long at a rate of 35 litres/minute at room temperature of 20ºC. What is the power required to maintain the flow if the pipe is horizontal and runs full of water. In order to answer this question it will be necessary to do the following:

- Calculate the Reynolds Number and decide if the flow is laminar or turbulent.
- Calculate the friction factor.
- Calculate the head loss due to friction.
- Calculate the power required to overcome the friction losses.

At each stage in these calculations the units should be checked. After all there is no point in getting to the end of a set of calculations only to find later that a units error has been made early in the calculations.

6.3.1 Calculation of Reynolds Number.

The calculation is best carried out using units of kilograms, metres and seconds. Fluid property tables often quote values in these units.

Reynolds Number $Re = \rho * v * d/\mu$ **eq(3)**

Where: ρ = Mass Density of water in kg/m^3 = 998.2 kg/m^3 at 20ºC
v = Mean Flow Velocity of the water in m/s
d = A Characteristic Length = pipe inside diameter in metres = 0.05 m
μ = Dynamic Viscosity of water in kg/(m·s) = 0.001002 kg/(m·s) at 20ºC

[**Note:** The middle dot between the m and the s, the units of dynamic viscosity are: kilogram per metre second, i.e. kg/(m·s) or $kg \cdot m^{-1} \cdot s^{-1}$ and NOT kg/ms as that would be kilogram per millisecond. Note also that the dynamic viscosity is sometimes quoted in units of $N \cdot s/m^2$ or mPa·s, i.e. milliPascal·second.]

We know that Reynolds number is dimensionless therefore the units of the RHS of equation **eq(3)** must cancel out completely. Check this out.

Check the units. The units on the RHS are: $kg \cdot m \cdot m/(m^3 \cdot s \cdot kg \cdot m^{-1} \cdot s^{-1})$ = $kg \cdot m \cdot m \cdot m \cdot s/(m^3 \cdot s \cdot kg)$ = $kg \cdot m^3 \cdot s/(m^3 \cdot s \cdot kg)$ which does cancel out completely ~~$kg \cdot m^3 \cdot s/(m^3 \cdot s \cdot kg)$~~, as required.

The mean flow velocity now requires to be calculated. The flow rate Q = mean flow velocity times the pipe flow area.

The flow rate $Q = v * A$ **eq(4)**

Where: A = pipe flow area = $\pi * d^2/4 = \pi * 0.05^2/4 = 0.00196\ m^2$

Rearranging **eq(4)** to give the mean flow velocity $v = Q/A$ **eq(5)**

Convert the units. The flow rate is given as 35 litres/minute. For consistent units the flow rate needs to be in cubic metres per second m^3/s. A conversion is required. A litre (symbol L) is equivalent to $10^{-3} m^3$. Therefore a flow rate Q = 35 L/minute = $35 * 10^{-3} m^3$/minute = $35 * 10^{-3}/60\ m^3$/second = 0.000583 m^3/s.

Check the units. The units on the RHS of **eq(5)** are: $m^3/(s \cdot m^2)$ which cancels ~~m^3~~/(s·~~m^2~~) to give the units of velocity m/s, as required.

From **eq(5)** the Mean Flow Velocity $v = Q/A$ = 0.000583/0.00196 = 0.297 m/s

Calculate the Reynolds number. From **eq(3)** $Re = \rho * v * d/\mu$ = 998.2∗0.297∗0.05/0.001002 = 14794 say 14800. Since Re is >4000 the flow is considered to be turbulent.

6.3.2 Calculation of the pipe Friction Factor *f*.

The friction factor, for turbulent flow in smooth pipes, can be calculated from the empirical relationship by Blasius in terms of the Moody friction factor.

Friction factor $f = 0.3164/Re^{1/4} = 0.3164*Re^{-1/4} = 0.3164*Re^{-0.25}$ **eq(6)**

Check the units. The number 0.3164 is a constant for smooth pipes and is dimensionless. We already know that the Reynolds number is dimensionless, we have just proved it above. The index 1/4 is a pure number and is also dimensionless, hence $Re^{1/4}$ is dimensionless. Therefore the friction factor is dimensionless.

From **eq(6**) the friction factor $f = 0.3164*14800^{-0.25} = 0.0287$

6.3.3 Calculation of the Head Loss due to friction h_f.

For a pipe running full of fluid the head loss can be calculated using the D'arcy equation.

Head loss $h_f = f*L*v^2/(2*g*d)$ **eq(7)**

Where: Friction factor $f = 0.0287$ as calculated above.
Pipe Length $L = 100$ m
Mean Flow Velocity $v = 0.297$ m/s as calculated above.
Pipe inside diameter $d = 0.05$ m
Acceleration due to gravity $g = 9.81\ \mathrm{m/s^2} = 9.81\ \mathrm{m{\cdot}s^{-2}}$

Check the units. The units of the RHS of the equation have to cancel to leave the head loss in metres. The 2 in the denominator is a pure number and is dimensionless, the friction factor is also dimensionless, we have just proved it above.

The units of the RHS of **eq(7)** become: $\mathrm{m{\cdot}m^2/(s^2{\cdot}m{\cdot}s^{-2}{\cdot}m)} = \mathrm{m}{\cdot}\cancel{\mathrm{m^2}}/(\cancel{\mathrm{s^2}}{\cdot}\cancel{\mathrm{m}}{\cdot}\cancel{\mathrm{s^{-2}}}{\cdot}\cancel{\mathrm{m}}) = \mathrm{m}$, as required.

The head Loss from **eq(7)**: $h_f = f*L*v^2/(2*g*d) = 0.0287*100*0.297^2/(2*9.81*0.05) = 0.26$ m.

6.3.4 Calculation of the Power requirements.

The power required to maintain the flow rate through the 100 m long pipe is:

Power $P = \rho*g*h_f*Q$ **eq(8)**

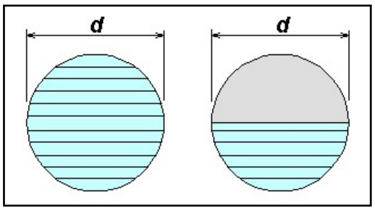
Check the units. The SI unit of power is the watt, symbol W. The watt is defined as a joule per second = J/s = newton metre/second = N·m/s. Therefore in base units the watt is a $\mathrm{kg{\cdot}m^2/s^3}$.

The units on the RHS of **eq(8)** are: $\mathrm{kg{\cdot}m{\cdot}m{\cdot}m^3/(m^3{\cdot}s^2{\cdot}s)} = \mathrm{kg{\cdot}m{\cdot}m}{\cdot}\cancel{\mathrm{m^3}}/(\cancel{\mathrm{m^3}}{\cdot}\mathrm{s^2{\cdot}s})$ which cancels to $\mathrm{kg{\cdot}m^2/s^3}$ = N·m/s = watts, as required.

Power required $P = 998.2*9.81*0.26*0.000583 = 1.48$ W.

Before leaving this example let us briefly remind ourselves that not all factors or coefficients are dimensionless. In the above example the friction factor (also known as the D'arcy coefficient) was dimensionless and this could be seen to be the case by the cancellation of the units. However, if the resulting head loss was calculated using the more general relationship for flow in a pipe or an open channel. This uses the Chézy coefficient, which for a drain has a value of about 55. Although just a number this coefficient does have units. Let us look at an example in some detail.

Example 6.4 Chézy coefficient.



For a drain pipe running full or half full of water it can be shown that the hydraulic mean depth is *d*/4 and the head loss in the pipe is given by:

$h_f = 4*L*v^2/(C^2*d)$ **eq(9)**

The variables *L*, *v* and *d* are as defined in **Example 6.3**. The 4 is a pure number and is dimensionless. *C* is a coefficient known as the Chézy coefficient. But is it dimensionless? We can check the units by cancellation and see.

We require the units of the RHS, of **eq(9)**, to cancel to give units of length to be consistent with the units of length (head loss) on the LHS. If we initially assume that the coefficient is dimensionless the units of **eq(9)** become: $m \cdot m^2/(s^2 \cdot m) = \cancel{m} \cdot m^2/(s^2 \cdot \cancel{m}) = m^2/s^2$. The units of the RHS do NOT cancel to give units of length. Clearly the Chézy coefficient is NOT dimensionless!

By inspection of equation **(9)**, the Chézy coefficient must have units of $\text{Length}^{1/2}/\text{Time}$ in order to allow the RHS of the equation to cancel to give units of length. <u>In SI units the Chézy coefficient must have units of $m^{1/2}/s$.</u>

We can check that this is correct by rearranging **eq(9)** in terms of the Chézy coefficient.

Chézy coefficient, $C = \{4 * L * v^2/(h_f * d)\}^{1/2}$ **eq(10)**

Check the units. The units of the RHS of **eq(10)** cancel to give: $[m \cdot \cancel{m^2}/(s^2 \cdot \cancel{m} \cdot \cancel{m})]^{1/2} = m^{1/2}/s$, as required.

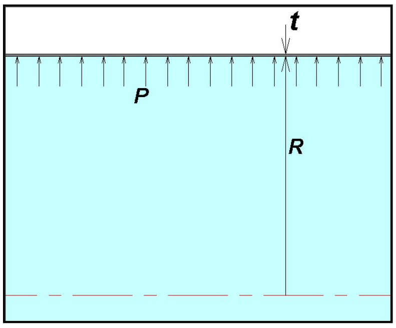
The lesson here is: Do not assume that any factors or coefficients are dimensionless unless you know they are dimensionless or can show that they are dimensionless. Coefficients, like the Chézy coefficient, that are not dimensionless should always be quoted with their units.

Calculate the result. Using a Chézy coefficient of 55 $m^{1/2}/s$ the results for the head loss over a length of 100 m becomes:

Head Loss **eq(9)**, $h_f = 4 * 100 * 0.297^2/(55^2 * 0.05) = 0.23$ m.

Therefore a fall or gradient of say 250 mm in 100 metres would be sufficient to maintain the flow rate.

Example 6.5 The Thickness of a Cylinder subjected to Internal Pressure.



This example is based on the equation presented by Cliff Matthews (ref. 2). It is a good example as it involves a sequence of terms in the denominator. He asks the question: 'If the maximum safe pressure *P* in an internally pressurised cylinder = $S * t/(R + 0.6 * t)$ where *t* = wall thickness, *R* = radius and *S* = allowable stress of the material, how thick does a cylinder of 2000 mm diameter need to be to resist a pressure of 10 bar(g) if it is constructed of a material with an allowable stress of 200 MPa?'

6.5.1 Rearrange the Equation.

We first need to rearrange the equation to give the wall thickness *t*. This is achieved in the following steps:

$P = S * t/(R + 0.6 * t)$
$P(R + 0.6 * t) = S * t$
$P * R + 0.6 * P * t = S * t$
$P * R = S * t - 0.6 * P * t$
$P * R = t(S - 0.6 * P)$
$t(S - 0.6 * P) = P * R$
$t = P * R/(S - 0.6 * P)$

Therefore the required wall thickness, $t = P * R/(S - 0.6 * P)$ **eq(11)**

6.5.2 Clarify the Equation (11).

At this point some clarification is required, as it is essential to know something about the equation.

Equations of this sort are used in several pressure vessel and boiler codes of practice (refs. 3, and 4) to calculate the cylinder wall thickness that will safely withstand the internal pressure to which the cylinder will be subjected. Some of these equations are based on diameter some are based on the inside dimensions (as assumed here) but in other cases the outside dimensions or even mean dimensions are used. In some cases a joint efficiency factor is also introduced.

Equation **(11)** is based upon thin shell theory but with a 0.6 simplification factor introduced so that the equation can be used with cylinders thicker than that normally allowed by thin shell theory. This factor is dimensionless and it is important to know this as not all factors or coefficients are non-dimensional as shown previously in **Example 6.4**.

6.5.3 Check the Units.

The LHS of equation **(11)** has units of length for the wall thickness. Therefore the RHS of the equation must cancel its units to give units of length also. This is essential. If the units do not cancel to give identical units on both sides of the equation then there is something wrong.

Suppose we do not know this and attempt to use the values and units as given: *P* = 10 bar(g), *S* = 200 MPa and *R* = 1000 mm. Equation **(11)** gives a wall thickness $t = 10*1000/(200 - 0.6*10) = 51.54$ but what are the units of the 51.54?

Let us put units into equation **(11)** and attempt to cancel them: *t* = Length = bar(g)·mm/(MPa - 0.6·bar(g)).

Right away we can see that there is a potential problem. The units of the groups of terms in the denominator have to be identical otherwise the values of the terms cannot be added algebraically, i.e. the *S* term and the 0.6**P* term must have identical units and we know from our knowledge of the equation that the 0.6 simplification factor is dimensionless and can be ignored in the units check. Therefore we can see immediately that the denominator units are not identical. *S* in MPa and *P* in bar(g) cannot be added algebraically to give any result that would allow the units of the denominator to cancel with the units of the numerator to leave units of length. Therefore the 51.54 result is garbage.

[**Note**: An experienced pressure vessel engineer would know that something was wrong from the magnitude of the result. 10 bar(g) is a relatively low pressure. A pressure vessel of 1000mm radius with an allowable strength of 200 MPa would not require to be 51.54 mm thick. A thickness of only 5 or 6 mm would do the job.]

The lesson here is: just because the values of the terms are quoted with particular units does not mean you blindly apply these values to an equation without first asking; is a units conversion required before I can safely apply the values to the equation?

6.5.4 Convert the Units.

There are two ways to proceed. Either the stress is converted to units of pressure or the pressure is converted to units of stress. From time immemorial engineers have calculated stresses as a Force/Length^2. So the logical way is to proceed is to convert the pressure to units of stress. Experience in the pressure vessel industry has shown that the most convenient unit for stress is the N/mm^2 for the following reasons:

- Many engineering drawings are scaled in mm.
- Several pressure vessel codes of practice have tables listing the material design strength in N/mm^2.
- Sensible engineering calculations should end up with stresses in tens or low hundreds of N/mm^2.
- The ultimate tensile strength of a typical carbon steel is 430 N/mm^2. If you get stress levels approaching this or higher then it is likely you have a potentially overstress problem.
- The S.I. units of stress is the N/m^2 (also known as the pascal). Unfortunately this is a very small unit for stress analysis purposes (a Newton is only about 0.225 lbf) and hence the MN/m^2 (or mega pascal MPa which equals a N/mm^2) is certain to be required in any engineering calculations.

Now we can deal with the units of stress as quoted. A MPa = MN/m^2 = N/mm^2. Therefore in this example the allowable stress *S* is simply = 200 N/mm^2.

Now we can deal with the units of pressure. The requirement is to get the pressure into units of N/mm^2 to be consistent with the units of stress. What is a bar(g) and what is its conversion to N/mm^2?

A bar is one of the special names decimally related to SI units, which may be used to denote pressure. As explained previously, in this context the symbol (g) stands for gauge pressure, i.e. above atmospheric pressure. It does NOT stand for the acceleration due to gravity nor does it stand for mass in grams. If the (g) had not been given it would be reasonable to assumed that the pressure in bar was gauge pressure. Occasionally the pressure may be quoted as absolute pressure in bar(a) i.e. above zero. Hence: absolute pressure = gauge pressure + atmospheric pressure.

A bar is almost an atmosphere. In meteorological terms the atmospheric pressure is often quoted as 1013.25 millibars at sea level, i.e. atmospheric pressure at sea level = 1.01325 bar absolute pressure. In the Imperial system this was approximately 14.7 lbf/in^2. Therefore a bar is approximately 14.5 lbf/in^2. In the SI system the bar = 0.1 MN/m^2 = 0.1 MPa = 0.1 N/mm^2, i.e. pressure in units of N/mm^2 = pressure in units of bar divided by 10. Therefore in this example the internal pressure *P* at a gauge pressure of 10 bar in N/mm^2 = 10/10 = 1 N/mm^2.

6.5.5 Check the Converted Units.

Embedding the converted units into the RHS of **eq(11)** gives: $N{\cdot}mm^{-2}{\cdot}mm/(N{\cdot}mm^{-2} - 0.6{\cdot}N{\cdot}mm^{-2})$. The 0.6 factor is dimensionless therefore the RHS units simplify to: $N{\cdot}mm^{-2}{\cdot}mm/(N{\cdot}mm^{-2} - N{\cdot}mm^{-2})$. The units of the terms within the ( ) of the denominator are now identical, as required.

The RHS further simplifies to: $N{\cdot}mm^{-2}{\cdot}mm/(N{\cdot}mm^{-2})$ and hence the units cancel to give ~~$N{\cdot}mm^{-2}$~~$\cdot mm/($~~$N{\cdot}mm^{-2}$~~$)$ mm, as required. Now we can calculate the correct answer.

6.5.6 Calculate the Correct Answer.

For an Internal Pressure *P* = 1 N/mm^2 (i.e. 10 bar), Allowable Stress *S* = 200 N/mm^2 (i.e. 200 MPa) and internal radius *R* = 1000 mm. Equation **(11)** gives a wall thickness *t* = 1∗1000/(200 - 0.6∗1) = 5.01 mm, the correct answer.

Before we leave this example. Let us consider if we could arrange to incorporate the units conversion into the equation so that the pressure is in bar and stress in N/mm^2.

6.5.7 Alternative Calculation incorporating the conversion into the equation, pressure in bar and stress in N/mm^2.

We know that pressure in units of N/mm^2 = pressure in units of bar ÷ 10 therefore we can write: the internal pressure *P* in N/mm^2 = pressure $P_{bar}/10$. The conversion factor of 10 has the units of: $bar/(N/mm^2)$, which can also be written as: $bar/(N{\cdot}mm^{-2}) = bar{\cdot}mm^2/N$.

Substituting $P_{bar}/10$ for *P* in **eq(11)** we have the wall thickness $t = P_{bar}{*}R/\{10(S - 0.6{*}P_{in\ bar}/10)\}$

This simplifies to $t = P_{bar}{*}R/(10{*}S - 0.6{*}P_{bar})$ **eq(12)**

Now we can check the units of **eq(12)**, remembering that the conversion factor of 10 has the units of $bar/(N/mm^2) = bar/(N{\cdot}mm^{-2}) = bar{\cdot}mm^2/N$ and will allow the allowable stress *S* to be in units of N/mm^2.

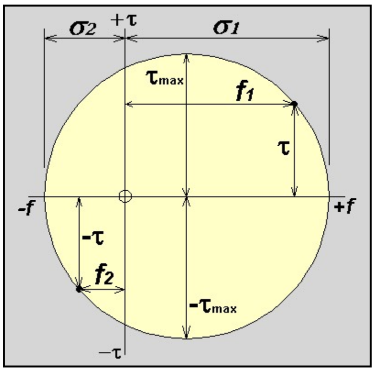
The RHS units of **eq(12)** = $bar{\cdot}mm/(bar{\cdot}mm^2{\cdot}N{\cdot}mm^{-2}/N - 0.6{\cdot}bar)$. The 0.6 factor is dimensionless as before therefore the RHS units simplifies to: $bar{\cdot}mm/(bar{\cdot}mm^2{\cdot}N{\cdot}mm^{-2}/N - bar)$. This simplifies by cancellation to bar·mm/(bar·~~$mm^2{\cdot}N{\cdot}mm^{-2}/N$~~ - bar) = bar·mm/(bar - bar) = bar·mm/(bar) and hence the units cancel ~~bar~~·mm/(~~bar~~) to give mm, as required.

Now we can recalculate the wall thickness using **eq(12)**. For an Internal Pressure P_{bar} = 10 bar, Allowable Stress *S* = 200 N/mm^2 (i.e. 200 MPa) and internal radius *R* = 1000 mm.

This gives a wall thickness *t* = 10∗1000/(10∗200 - 0.6∗10) = 5.01mm. The correct answer as before.

[**Note**: although this alternative calculation, by incorporating the conversion of units within the equation, does work personally I prefer to do the conversion of units first then do the final calculation with the original equation.]

Example 6.6 Calculation of Principal Stresses.



This example introduces the well known Mohr circle equations. These equations comprise of groups of terms and have a square root as part of one of the groups.

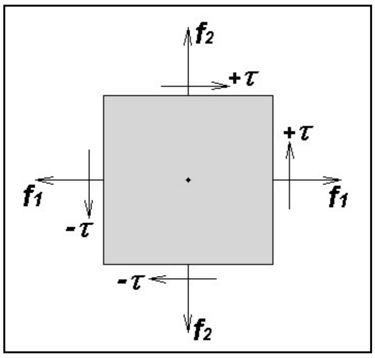
The following stresses have been calculated at a point in a structure. Direct stress in the longitudinal direction = 55 N/mm^2. Direct stress in the transverse direction = -15 N/mm^2. The complimentary shear stress in each of these directions = 30 N/mm^2. It is required to calculate the two principal stresses and the maximum shear stress at the point.

The two principal stresses σ_1 and σ_2 are given by the following equation:

$$\sigma_{1,2} = 0.5{*}(f_1 + f_2) \pm 0.5{*}\sqrt{\{(f_1 - f_2)^2 + 4{*}\tau^2\}} = 0.5{*}(f_1 + f_2) \pm 0.5{*}\{(f_1 - f_2)^2 + 4{*}\tau^2\}^{1/2} \qquad \textbf{eq(13)}$$

and the maximum shear stress becomes:

$\tau_{max} = \pm (\sigma_1 - \sigma_2)/2$ **eq(14)**



where: σ_1 = first principal stress.
σ_2 = second principal stress.
f_1 = direct stress on face 1 (taken to be the longitudinal direction).
f_2 = direct stress on face 2 (taken to be the transverse direction).
τ = shear stress on face 1 and 2.
τ_{max} = maximum shear stress.

Direct and principal stresses are +ve tensile, -ve compression. Shear stresses are ± by the rules of complementary shear.

6.6.1 Check the Units.

The LHS of equations **(13)** and **(14)** requires the units of stress, i.e. N/mm^2. Therefore the groups of terms on the RHS of these equations must cancel to give N/mm^2 also. Check this out.

(1) With reference to equation (13). The variables f_1, f_2 and τ must have identical units otherwise there is no way the terms of the groups $(f_1 + f_2)$, $(f_1 - f_2)^2$ and $4*\tau^2$ can be added algebraically to give any sensible answers.

The units of f_1, f_2, and τ are all N/mm^2 therefore groups $(f_1 + f_2)$, $(f_1 - f_2)^2$, and $4*\tau^2$ can be added algebraically, therefore we can proceed to check the units in detail.

The units of group $0.5*(f_1 + f_2)$ and $0.5*\{(f_1 - f_2)^2 + 4*\tau^2\}^{1/2}$ must now be checked. The numbers 0.5 and 4 are dimensionless and can be ignored from the units check. The indexes 2 and 1/2 (for square root) are dimensionless pure numbers and have no units but these need to be kept in the units check otherwise the units will not cancel correctly.

The units of the first group $0.5*(f_1 + f_2)$ are: $(N/mm^2 + N/mm^2)$. The units of the terms are identical which is correct. Therefore the units of the group $0.5*(f_1 + f_2)$ is N/mm^2, as required.

The units of the second group $0.5*\{(f_1 - f_2)^2 + 4*\tau^2\}^{1/2}$ are: $\{(N/mm^2 - N/mm^2)^2 + N^2/mm^4\}^{1/2}$ which can be written as $\{(N^2/mm^4) + N^2/mm^4\}^{1/2}$. Cancelling the square root leaves us with: $\{(N/mm^2) + N/mm^2\}$. The units of the terms are identical which is correct.
Therefore the units of the group $0.5*\{(f_1 - f_2)^2 + 4*\tau^2\}^{1/2}$ is N/mm^2, as required.

The units of all groups of equation **(13)** are identical and N/mm^2, as required.

(2) With reference to equation (14), the variables σ_1 and σ_2 must have identical units so that the group $(\sigma_1 - \sigma_2)$ can be added algebraically. The 2 in the denominator is dimensionless and has no units and can be ignored from the units check. The units of σ_1 and σ_2 from equation **(13)** are identical and shown to be N/mm^2, therefore the units of equation **(14)** are N/mm^2, as required.

All units being correct we can proceed with the calculation.

6.6.2 Calculate the results. For the given stresses: Direct stress, $f_1 = 55\ N/mm^2$, Direct stress, $f_2 = -15\ N/mm^2$ and Shear stress, $\tau = 30\ N/mm^2$

The principal stresses are, from equation **(13)**:

$$\sigma_1 = 0.5*(55 +[-15]) + 0.5*\{(55 - [-15])^2 + 4*30^2\}^{1/2} = 66.1\ N/mm^2$$

$$\sigma_2 = 0.5*(55 +[-15]) - 0.5*\{(55 - [-15])^2 + 4*30^2\}^{1/2} = -26.1\ N/mm^2$$

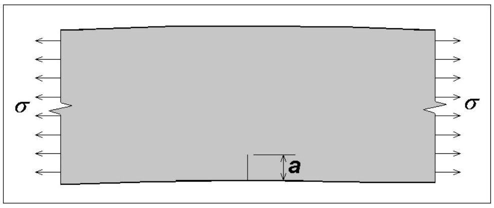
and the maximum shear stress, from equation **(14)** is:

$$\tau_{max} = (66.1 - [-26.1])/2 = 46.1\ N/mm^2$$

The lesson here is: It is important to check individual groups of an equation. This is essential to ensure that a consistent set of units is being used and that identical units are obtained before attempting to add groups algebraically.

Example 6.7 Fast Fracture.

This example introduces the concept of fast fracture, which has terms with slightly unusual units. A pressure vessel example will be chosen but the concept is applicable any structure where linear-elastic fracture mechanics applies. This example will use the usual SI units for the material fracture toughness of $MN/m^{3/2}$ i.e. mega newton per metre to the power 1.5.



[**Note**: In past practice several different units for fracture toughness have been in use (and might still be in use), e.g. $MPa \cdot m^{1/2}$ i.e. mega pascal·metre to the power 0.5; $MPa \cdot mm^{1/2}$ i.e. mega pascal·millimetre to the power 0.5; $N/mm^{3/2}$ i.e. newton per millimetre to the power 1.5; and $KSI \cdot inch^{1/2}$ i.e. (1000 $lbf/inch^2$)·inch to the power 0.5. Whatever units are being used particular care has to be taken to ensure that the units are consistent.]

A cylindrical ferritic steel pressure vessel is subjected to an internal pressure that will stress the wall with a hoop stress of 150 N/mm^2. The wall thickness is 120 mm. The vessel is designed to a recognised pressure vessel code so is known to be safe from plastic collapse. However, during one of the operating conditions a low temperature (< 0°C) may be experienced such that the fracture toughness of the material could be as low as 50 $MN/m^{3/2}$ and hence there may be a risk of failure by brittle fracture. It is required to establish whether the pressure vessel will 'leak before it will break' or will it fast fracture? The decision one way or the other is important as a 'leak before it will break' would give a warning that the wall had been breached but if the critical crack size was less than the vessel thickness then there is a risk of sudden catastrophic failure.

The potential for fast fracture, to occur in a structure, is when the crack stress intensity factor is > or = the material fracture toughness, i.e. the onset of fast fracture is assumed to occur in a structure when $K = K_c$

Where: K is the crack stress intensity factor = $C*\sigma*\sqrt{(\pi*a)} = C*\sigma*(\pi*a)^{1/2}$

σ is the loading stress remote from the crack tip; in this application this is the hoop (circumferential) stress in the cylinder wall due to internal pressure.

a is the crack size (crack depth) of a surface crack.

C is a dimensionless correction factor, that depends on the geometry of the structure, taken as 1.12 for this application.

K_c is a material property called the fracture toughness, sometimes called the 'critical stress intensity factor',

$K_c = \sqrt{\{E*G_c/(1-\mu^2)\}} = \{E*G_c/(1-\mu^2)\}^{1/2}$ assuming plane strain conditions.

E is a material property, the modulus of elasticity (Young's modulus) with units of stress.

μ is the material property Poisson's ratio. It is dimensionless and has no units.

G_c is a material property called the toughness, sometimes called the 'critical strain energy release rate' = $K_c^2*(1-\mu^2)/E$

For the onset of fast fracture $K = K_c = C*\sigma*(\pi*a)^{1/2}$ **eq(15)**

6.7.1 Rearrange the Equation.

We need to rearrange equation **(15)** to give the length at which a crack would reach a critical crack size.

$C*\sigma*(\pi*a)^{1/2} = K_c$

$C^2*\sigma^2*(\pi*a) = K_c^2$

$\pi*a = K_c^2/(C^2*\sigma^2)$

$a = K_c^2/(\pi*C^2*\sigma^2)$

Therefore the critical crack size $a_{crit} = K_c^2/(\pi*C^2*\sigma^2)$ **eq(16)**

6.7.2 Clarify the Equation.

At this point some clarification of the terms is required and it helps if you know something about the equation.

The factor C is dimensionless and π is of course also dimensionless.

By inspection of **eq(15)** the units of K must be: Stress.Length$^{1/2}$ or Force.Length$^{-3/2}$ = Force/Length$^{3/2}$. If we work with units of force in mega newton MN and length in metres m. The units of the crack stress intensity factor K becomes $MN/m^{3/2}$. The units of fracture toughness K_c must be the same as the units for the crack stress intensity factor K. This must be the case since we are equating $K = K_c$ as the onset of fast fracture in equation **(15)**. Check out the units of K_c.

6.7.3 Check the Units of K_c.

$$K_c = \{E * G_c/(1-\mu^2)\}^{1/2} \qquad \textbf{eq(17)}$$

The units of E, the modulus of elasticity, is Stress i.e. Force/Length^2
Poisson's ratio μ is dimensionless and has no units, and the group of terms $(1-\mu^2)$ is also dimensionless.

The units of G_c is Energy/Length^2. The units of energy is Force·Length. Hence the units of G_c is Force/Length.

The units of K_c in equation **(17)** are therefore = $\{\text{Force}\cdot\text{Force}/(\text{Length}^2\cdot\text{Length})\}^{1/2}$ = $\{\text{Force}^2/\text{Length}^3\}^{1/2}$ = Force/Length$^{3/2}$ = MN/m$^{3/2}$ as required.

[**Note**: Sometimes the fracture toughness K_c is quoted in the alternative units of: Stress·Length$^{1/2}$ with stress in mega pascal and length in metres, the units of K_c become MPa·m$^{1/2}$ i.e. mega pascal·metre to the power 0.5. Since a MPa = MN/m^2 i.e. a mega newton per metre squared, the units of Stress·Length$^{1/2}$ become: MN·m^{-2}·m$^{1/2}$ and cancel MN·~~m^{-2}~~·~~m$^{1/2}$~~ to give units of MN·m$^{-3/2}$ = MN/m$^{3/2}$ = Force/Length$^{3/2}$, as required for consistency.]

6.7.4 Check the Units of a_{crit}.

The RHS of equation **(16)** must cancel to give units of length.

The units are: $(\text{MN}\cdot\text{m}^{-3/2})^2/(\text{MN/m}^2)^2 = (\text{MN}^2\cdot\text{m}^{-3})/(\text{MN}^2/\text{m}^4)$.
These cancels as follows: = (~~MN^2~~·m^{-3})/(~~MN^2~~/m^4) = m^{-3}/(1/m^4) = ~~m^{-3}/m^{-4}~~ = 1/m^{-1} = m, as required.

6.7.5 Calculate the Critical Crack Size.

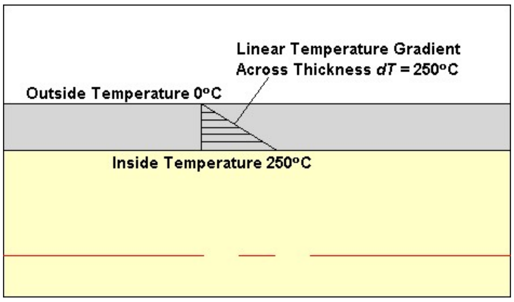
For a stress of σ = 150 N/mm^2 (i.e. 150 MN/m^2) and fracture toughness K_c of 50 MN·m$^{-3/2}$. Equation **(16)** gives a critical crack size $a_{crit} = 50^2/(\pi * 1.12^2 * 150^2)$ = 0.0282 m. i.e. 28.2 mm. The critical crack size is < the cylinder thickness of 120 mm. Therefore at this particular operating condition, should a surface crack approach 28.2 mm, there is a risk of fast fracture occurring!

[**Note**: The crack size for sub-surface flaws is $2a$. i.e. the crack size would need to approach 56.4 mm before there would be a risk of fast fracture. However, if the flaw should break through to the surface then the flaw would be re-categorised as a surface crack with a_{crit} = 28.2 mm.]

[**Note**: The above example uses linear-elastic fracture mechanics and plane strain assumptions. However, for many applications, where thin ductile materials are in use, elastic-plastic fracture mechanics and plane stress assumptions apply. In these cases the use of the path independent J-Integral or the Crack Tip Opening Displacement method of analysis would be more applicable.]

6.8 A Thermal Stress Problem.

This example introduces the use of temperature and material properties. A long thin cylinder manufactured from ferritic steel is subjected to a temperature difference of 250°C across the cylinder thickness. It is requires to calculate the magnitude and sign of the thermal stresses on the surface of the cylinder, remote from the ends, and assuming the temperature gradient across the thickness is linear.



[**Note**: It is important to note that the analysis applies remote from the ends. At the ends there is often some additional bending of the shell. It can be shown that at a free end the stresses are actually some 25% higher than that remote from the ends.]

The temperature difference will induce equal longitudinal and circumferential (hoop) thermal bending stresses at the surface of the cylinder. The magnitude of these bending stresses is given by the following equation:

$$\sigma_L = \sigma_H = \pm\, 0.5 * E * \alpha * dT/(1-\mu) \qquad \textbf{eq(18)}$$

Where: σ_L is the longitudinal thermal bending stress at the surface of the cylinder.

σ_H is the circumferential (hoop) thermal bending stress at the surface of the cylinder.
E is the material property the modulus of elasticity (Young's Modulus). It has units of stress and varies with temperature. For a ferritic steel at 250°C, E = 195000 N/mm^2.
α is the material property the coefficient of linear thermal expansion. It has units of 1/one interval (degree) of temperature change and varies slightly with temperature.
For ferritic steel the mean (average) coefficient of linear thermal expansion from 20°C to 250°C is α = 13.0×-6/°C = 0.000013/°C.
μ is the material property Poisson's ratio = 0.3 for most metals. It is dimensionless and has no units.
dT is the temperature difference across the cylinder thickness = 250°C.

[**Note**: The thermodynamic temperature interval of degree Celsius °C (or degree centigrade) is identical to the SI temperature interval of thermodynamic temperature the degree kelvin (symbol K). In Imperial units the temperature interval of degree Fahrenheit °F is 5/9 of a degree Celsius °C, i.e. in the above example the temperature difference *dT* of 250°C = 450°F and the coefficient of linear thermal expansion = 0.00000722/°F. The modulus of elasticity is sometimes quoted as a giga newton per metre squared, i.e. in the above example E = 195 GN/m^2 = 195000 N/mm^2.]

[**Note**: The coefficient of linear thermal expansion may occasionally be given as a Length/Length·degree of temperature change, i.e α = mm/mm/°C so as to make it clear that linear expansion is being considered. However, what is actually meant here is: mm/(mm·°C) or (mm/mm)/°C. Where the length in the numerator is the thermal expansion in one interval, i.e. degree, of temperature change and the length in the denominator is the length over which the expansion applies. e.g. for the ferritic steel above, the thermal expansion is 0.000013 mm over a length of 1 mm when the temperature changes by 1 degree Celsius. As the units of length cancel out ~~mm~~/(~~mm~~·°C) or (~~mm~~/~~mm~~)/°C the units of coefficient of linear thermal expansion are: 1/one degree of temperature change. As the coefficient varies slightly with temperature it is often quoted as a mean or average temperature from room temperature (20°C) to the indicated temperature.]

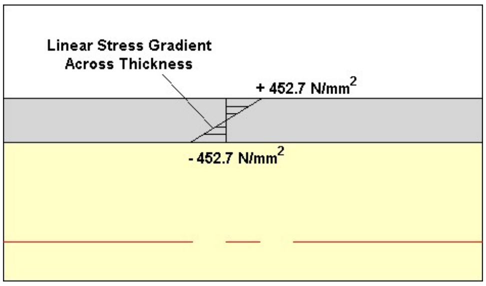
6.8.1 Check the Units.

The LHS of equation **(18)** requires units of stress such as N/mm^2 = N·mm^{-2}. Therefore the groups of terms on the RHS of the equation must cancel to give units of stress also.

The 0.5 is a dimensionless number. Poisson's ratio μ being a ratio is also dimensionless and hence the denominator group (1 - μ) is dimensionless.

The units of the RHS of equation **(18)** are therefore: N·mm^{-2}·°C/°C which cancels N·mm^{-2}·~~°C~~/~~°C~~ to give N/mm^2, as required.

6.8.2 Calculate the Thermal Stresses

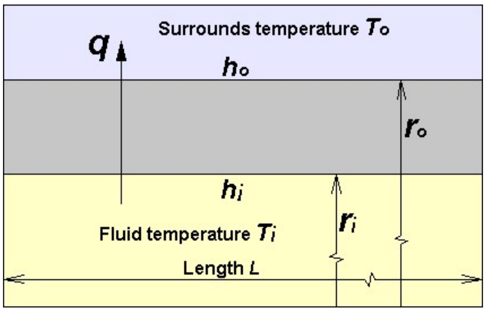


Equation **(18)** gives stresses of:

$$\sigma_L = \sigma_H = \pm 0.5*195000*0.000013*250/(1 - 0.3) = \pm 452.7 \text{ N/mm}^2$$

The stresses are positive tensile on the *outside* surface if the temperature of the cylinder is higher on the *inside* surface.

6.9 Heat Transfer by Conduction and Convection.



This example introduces the use of thermal conductivity and heat transfer coefficients. It is required to calculate the heat transfer rate across the thickness of a 1.5 metre length of carbon steel pipe when subject to a temperature difference of 150°C. The dimensions the pipe are: 609.6 mm outside diameter and wall thickness 12.7 mm. The heat transfer coefficients between the fluids and the pipe wall are: 20 W/m^2·°C and 2.5 W/m^2·°C on the inside and outside surfaces respectively.

The heat transfer rate through the wall thickness is given by the following equation:

$$q = C*(T_i - T_o) = C*dT \qquad \textbf{eq(19)}$$

Where: q is the heat transfer rate (heat flow rate) in joule/second, J/s = N·m/s = watts W.
T_i is the temperature of the fluid inside the pipe.
T_o is the temperature of the surrounds outside the pipe.

dT is the temperature difference = $(T_i - T_o)$ = 150°C
C is the thermal conductance, i.e. the heat transfer rate per unit temperature difference,
for a pipe $C = 2*\pi*L/\{1/(h_i*r_i) + \ln(r_o/r_i)/\lambda + 1/(h_o*r_o)\}$
L is the length of pipe = 1.5 m
r_o is the pipe outside radius = 304.8 mm = 0.3048 m
r_i is the pipe inside radius = 292.1 mm = 0.2921 m
ln is logarithm to the base e.
λ is the thermal conductivity of the pipe material. For steel λ = 49.8 W/(m·°C)
h_o is the heat transfer coefficient on the outside surface of the pipe = 2.5 W/(m^2·°C)
h_i is the heat transfer coefficient on the inside surface of the pipe = 20 W/(m^2·°C)

6.9.1 Check the Units.

The LHS of equation **(19)** requires units of heat flow which is the watt (symbol W). Therefore the groups of terms on the RHS of the equation must cancel to give units of watts also.

The first step is to check the units of thermal conductance C. Since the temperature difference dT has units of °C, the units of thermal conductance must have units of W/°C in order that the units on the RHS of equation **(19)** cancel to give watts.

$$C = 2*\pi*L/\{1/(h_i*r_i) + \ln(r_o/r_i)/\lambda + 1/(h_o*r_o)\} \qquad \textbf{eq(20)}$$

The RHS of equation **(20)** must cancel to give W/°C. The terms 2 and π are dimensionless and have no units. The length L has units of Length in metres m. We must check to see if the units of the groups of terms within the brackets { } are identical. Check this out.

$1/(h_i*r_i)$ = 1/(W·m/(m^2·°C)) = 1/(W·~~m~~/(~~m~~^{2}·°C)) = 1/(W/(m·°C)) = m·°C/W

$\ln(r_o/r_i)$ is dimensionless and has no units; it is convenient to replace it with a 1 so that $\ln(r_o/r_i)/\lambda = 1/\lambda$ in the units check. Therefore: $1/\lambda$ = 1/(W/(m·°C)) = m·°C/W

$1/(h_o*r_o)$ = 1/(W·m/(m^2·°C)) = 1/(W·~~m~~/(~~m~~^{2}·°C)) = 1/(W/(m·°C)) = m·°C/W

The units of all the groups of terms in the brackets { } of **eq(20)** are identical, we can proceed.

Check units of thermal conductance *C*.
The units of C in equation **(20)** become: m/{m·°C/W} = m·W/(m·°C) = ~~m~~·W/(~~m~~·°C) = W/°C, as required.

Check units of the heat transfer rate *q*.
The units of q equation **(19)** are: W·°C/°C = W·~~°C~~/~~°C~~ = W, watts, as required.

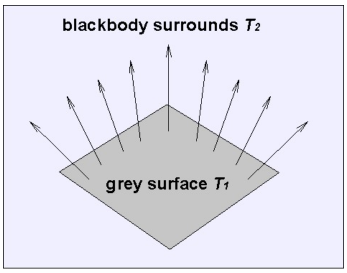
6.9.2 Calculate the Heat Flow Rate

The thermal conductance C, **eq(20)** = 2*π*1.5/{1/(20*0.2921) + ln(0.3048/0.2921)/49.8 +1/(2.5*0.3048)}
C = 6.35 W/°C

Heat transfer rate q, **eq(19)** = 6.35*150 = 952.5 W i.e almost a kW of heat transfer from a 1.5 metre length of pipe.

6.10 Heat Transfer by Radiation.

This example introduces heat transfer which requires absolute temperatures. The rate of heat transfer by radiation depends on the 4th power of the absolute temperature. It is required to calculate the heat transfer rate from a grey surface of area 3 m^2 to the blackbody surroundings. The temperature of the surface is 150°C and the temperature of the surroundings is 20°C. The emissivity of the grey surface is 0.9, i.e. it emits 90% radiant energy of a blackbody.



The radiation heat transfer rate is given by the following equation:

$$q = \varepsilon*\sigma*(T_1^4 - T_2^4)*A \qquad \textbf{eq(21)}$$

Where: q is the radiation heat transfer rate (heat flow rate) in watts

T_1 is the absolute temperature of the grey surface in kelvin units K
T_2 is the absolute temperature of the surroundings in kelvin units K
A is the surface area of the grey surface = 3 m^2
ε is the emissivity of the grey surface.
σ is the Stefan-Boltzmann constant = 5.67×10^{-8} W/($m^2 \cdot K^4$).
[**Note**: Not to be confused with the Boltzmann constant, which is something else entirely.]

6.10.1 Check the Units

The LHS of **eq(21)** requires units of heat flow which is the watt (symbol W). Therefore the groups of terms on the RHS of the equation must cancel to give units of W also.

The emissivity ε is an effectiveness ratio in emitting thermal energy. It is dimensionless and has no units. The Stefan-Boltzmann constant has units of W/($m^2 \cdot K^4$). The absolute temperatures T_1 and T_2 must have SI units of kelvin K. The group ($K^4 - K^4$) has units of K^4. The units of surface area is m^2.

Therefore the units of the RHS of **eq(21)** are: ($W \cdot K^4 \cdot m^2$)/($m^2 \cdot K^4$) which cancel (W·~~K^4~~·~~m^2~~)/(~~m^2~~·~~K^4~~) to W watts, as required.

6.10.2 Calculate the Heat Flow Rate.

Before we can calculate the heat flow rate we need to get the temperature units converted into absolute units of kelvin.

[**Note**: at one time degree kelvin °K was permitted, but SI have now decided that the use of the term degree is to be omitted.]

Temperature in kelvin = temperature in degrees Celsius + 273

Therefore, converted temperatures are: T_1 = 150°C + 273 = 423 K.
T_2 = 20°C + 273 = 293 K.

[**Note**: If degrees Fahrenheit °F was being used then it is necessary to convert to degrees Rankine °R. Temperature in degrees Rankine = temperature in degrees Fahrenheit + 460.]

The radiation heat transfer rate, equation **(21)**, $q = 0.9 * 5.67 \times 10^{-8} * (423^4 - 293^4) * 3 = 3772.9$ W
i.e. 3.77 kW from the 3 m^2 surface to the surroundings.

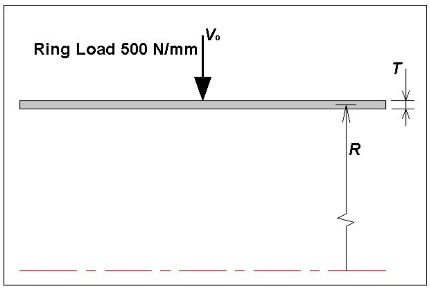
[**Note**: If non-absolute temperature units had been used in the above equation e.g. °C or °F the result would have been completely wrong. Suppose we had used °C, the result of **eq(21)** would be $q = 0.9 * 5.67 \times 10^{-8}(150^4 - 20^4) * 3 = 77.5$, which we know must be wrong as the units ($W \cdot °C^4 \cdot m^2$)/($m^2 \cdot K^4$) cancel to give $W \cdot °C^4/K^4$. Although °C and K have the same interval (increment) of temperature we cannot just cancel them as °C is NOT an absolute temperature scale where as the Kelvin scale is. 20°C (293 K) is NOT twice as hot as 10°C (283 K) but 20 K is twice as hot as 10 K.]

The lesson here is: Always check to see whether temperature intervals can be applied to an equation or is it essential, as in this example, to apply absolute temperatures and hence essential to use an absolute temperature scale like the Kelvin scale (or the Rankine scale).

6.11 An Axisymmetric Shell Junction Problem.

There are many shell structures, such as boilers and pressure vessels, which have axisymmetric junctions. These structures, when subjected to loading, will have discontinuities at and close to the junction, see Whyte reference 5 for a discussion and method of analysis. This example introduces an axisymmetric problem and shows that some care is required to ensure that the units of the 'stress resultants' are correct.

A long steel cylinder 750 mm mean radius and 20 mm thick is subjected to an inward radial ring load of V_o = 500 N/mm circumference. It is required to obtain the discontinuity stresses, in the cylinder, at the ring load (the discontinuity junction). The discontinuity forces and moments (sometimes called 'stress resultants') can be readily calculated from a junction analysis program such as AJAP-1 (ref. 5).



The discontinuity stresses are calculated from the following equations:

Meridional membrane stress, *SL* = *N1*/*T* **eq(22)**

Hoop membrane stress, *SH* = *N2*/*T* **eq(23)**

Meridional bending stress, *SBL* = ± 6∗*M1*/T^2 **eq(24)**

Hoop bending stress, *SBH* = ± 6∗*M2*/T^2 **eq(25)**

Average shear stress, *SV* = *V*/*T* **eq(26)**

By the sign convention used in reference 5. Tensile stresses are positive +ve. Compressive stresses are negative –ve. Positive *SL* or *SH* indicates tensile stress a negative sign indicates compression stress. The sign of *SBL* or *SBH* if negative it indicates compression stress on the *outside* surface and a tensile stress on the *inside* surface. *SV* is positive if shear force *V* is +ve outwards.

Where: *N1*, *N2*, *M1*, *M2*, and *V* are the forces and moments acting at a section, across the thickness, of the cylinder as a result of the applied load. They are sometimes called stress resultants but note they do NOT have units of stress but have units of force or moment per unit length of circumference.

N1 is the meridional (longitudinal) membrane force per unit length of circumference. In this example there are no longitudinal forces therefore *N1* = 0

N2 is the circumferential (or hoop) membrane force per unit length of circumference, +ve tensile.

N2 = *Y*∗*E*∗*T*/*R* **eq(27)**

M1 is the meridional bending moment per unit length of circumference, +ve if compression on the outside surface.

M2 is the circumferential (hoop) bending moment per unit length of circumference, +ve if compression on the outside surface.

M2 = μ∗*M1* **eq(28)**

V is the shear force per unit length of circumference, +ve outward.

T is the cylinder thickness = 20 mm
R is the mean radius of the cylinder = 750 mm
Y is the radial deflection, +ve outwards
E is the modulus of elasticity (Young's modulus) = 209000 N/mm^2 for Carbon steel at room temperature.
μ is Poisson's ratio = 0.3 for steel.

[**Note**: It is perfectly valid to formulate these equations as total forces and moments of the circumference. Equally valid is to formulate the equations as forces and moments per unit angle of the circumference, e.g. forces and moments per radian of the circumference or forces and moments per degree of the circumference. It is essential to check which definition of units has been used in the formulation of the equations. In this text the equations are formulated as force or moment per unit length of circumference since that is how they are defined in references such as Hetényi, ref. 6, Timoshenko, ref. 7, Flügge, ref. 8, and Roark, ref. 9.]

6.11.1 Check the Units of the stress resultants.

The LHS of **eq(27)** requires units of force per unit length of circumference. Therefore RHS of the equation must cancel to give units of force per unit length of circumference also. Check this out.

The units of the RHS of **eq(27)** are: mm·N·mm/(mm^2·mm) = ~~mm~~·N·~~mm~~/(~~mm^2~~·mm) = N/mm, as required.

The LHS of **eq(28)** requires units of moment per unit length of circumference. Therefore on the RHS of the equation must cancel to give units of moment per unit length of circumference also.

Poisson's ratio μ is dimensionless and has no units. Therefore the units of *M2* are the same as *M1* i.e. moment per unit length of circumference = N·mm/mm, as required.

[**Note**: it is common practice not to cancel out the length when checking these units so that it is clear that the numerator is a moment and the denominator is a unit length of the circumference.]

We can now check the stress units.

6.11.2 Check the Units of the stresses.

The LHS of equations **(22)** to **(26)** requires units of stress i.e. N/mm^2.

The RHS of **eq(22)**, **eq(23)** and **eq(26)** is (N/mm)/mm = $(N{\cdot}mm^{-1})/mm = N{\cdot}mm^{-1}{\cdot}mm^{-1} = N{\cdot}mm^{-2} = N/mm^2$, as required.

The 6 in equations **(24)** and **(25)** is dimensionless with no units.

The RHS of **eq(24)** and **eq(25)** is $(N{\cdot}mm/mm)/mm^2$ = (N·~~mm~~/~~mm~~)/mm^2) = N/mm^2, as required.

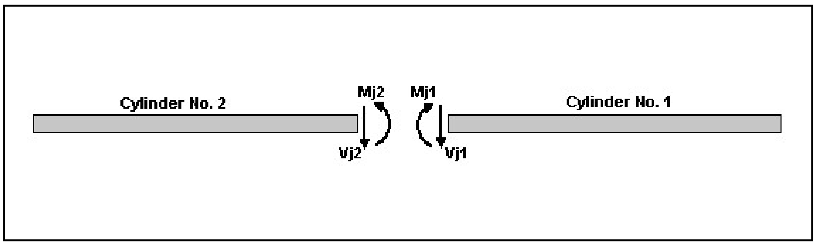
All the stress units are identical, as required.

6.11.3 Calculate the Stresses in the Cylinder.

The discontinuity in this example can be analysed as a two-cylinder junction problem where the junction is at the applied load V_o. The junction results can be obtained from a discontinuity analysis program such as AJAP-1 (ref. 5). The junction results from AJAP-1 are:

Component No. 1 Cylinder
Mj1= 11910.09 Nmm/mm clockwise
Vj1= 250 N/mm inward

Component No. 2 Cylinder
Mj2= 11910.09 Nmm/mm anti-clockwise
Vj2= 250 N/mm inward



Radial deflection *Y* at the junction i.e. under the applied load = -0.3530867 mm inward.

Because the geometry, loading and material properties are symmetric it is only necessary to analyse one of the cylinders, say cylinder No. 1, to get the results.

Stress results for cylinder No. 1 at the junction.

SL = 0 as *N1* is zero since there are no longitudinal forces.

From **eq(27)**, hoop membrane force $N2 = Y{*}E{*}T/R$ = -0.3530867∗209000∗20/750 = -1967.87 N/mm

From **eq(23)**, hoop membrane stress $SH = N2/T$ = -1967.87/20 = -98.39 N/mm^2 compression stress.

Meridional bending moment $M1 = Mj1$ = 11910.09 Nmm/mm +ve as an clockwise moment at the junction produces compression on the outside surface.

From **eq(24)**, $SBL = \pm 6{*}M1/T^2 = \pm 6{*}11910.09/20^2 = \pm 178.65\ N/mm^2$. Since *M1* is +ve the stress will be compressive on the outside surface of the cylinder and tensile on the inside surface.

From **eq(28)**, hoop bending moment $M2 = \mu{*}M1$ = 0.3∗11910.09 = 3573.027 Nmm/mm

From **eq(25)**, $SBH = \pm 6{*}M2/T^2 = \pm 6{*}3573.026/20^2 = \pm 53.59\ N/mm^2$. Since *M2* is +ve the stress will be compressive on the outside surface of the cylinder and tensile on the inside surface.

The shear force *V* = -*Vj1* = -250 N/mm the –ve sign indicates that V is inward.

From **eq(26)**, average shear stress $SV = V/T$ = -250/20 = -12.5 N/mm^2 inward.

Therefore the stresses in the cylinder at the applied load i.e. at the junction can be summarised:

SL = 0
SH = -98.39 N/mm^2

SBL = ± 178.65 N/mm^2 compression on the outside surface
SBH = ± 53.59 N/mm^2 compression on the outside surface
SV = -12.5 N/mm^2

To find the maximum stress it is necessary to sum the membrane and bending stresses this must be done algebraically, i.e the correct sign must be taken into account.

The meridional membrane + bending stress, at the outside surface, SLo = $SL + SBL$
= 0 + (-178.65) = -178.65 N/mm^2

The meridional membrane + bending stress, at the inside surface, SLi = $SL + SBL$
= 0 + 178.65 = 178.65 N/mm^2

The hoop membrane + bending stress, at the outside surface, $SHo = SH + SBH$
= -98.39 + (-53.59) = -151.98 N/mm^2

The hoop membrane + bending stress, at the inside surface, $SHi = SH + SBH$
= -98.39 + 53.59 = -44.8 N/mm^2

[**Note**: In this case the maximum stress is due to the meridional bending stress at the junction. However, in general it is possible to have cases where the maximum stress may be slightly away from the junction.]

6.12 A Thin Shell Theory Problem.

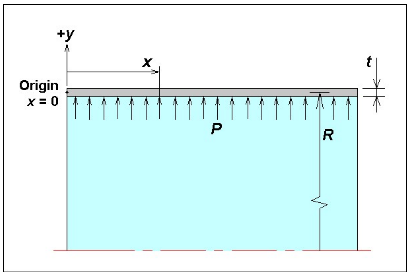
The previous example makes use of thin shell theory detailed in many references such as: Hetényi, ref. 6, Timoshenko, ref. 7, Flügge, ref. 8, and Roark, ref. 9. It is worth having a look at the units of these equations and the way the theory of these shells is applied.

Taking a case of a thin shell of uniform thickness. The governing equation can be shown to be the following 4th order ordinary differential equation:

$$a_0 * d^4y/dx^4 + a_4 * y = f(x) \qquad \textbf{eq(29)}$$

Where: y is the dependent variable, usually a deflection or a shear force per unit length of circumference.
x is the independent variable, usually a distance or an angle along the shell from the origin.
a_0 and a_4 are constants (parameters) related to the geometry and material properties of the shell structure.
$f(x)$ is the loading intensity, which could be zero, uniform or a function of x.

6.12.1 Cylindrical Shell.



For a thin cylindrical shell of uniform thickness and subjected to a uniform internal pressure the differential equation **eq(29)** can be shown to be:

$$D * d^4y/dx^4 + 4 * \beta^4 * D * y = P$$

and after rearranging:

$$d^4y/dx^4 + 4 * \beta^4 * y = P/D \qquad \textbf{eq(30)}$$

Where: y is the radial deflection of the shell with units of Length, +ve upwards (outwards).
x is the distance from the origin to a section across the thickness of the cylinder, with units of Length.
D and $4 * \beta^4$ are constants (parameters) related to the geometry and material properties of the shell structure.
D is the flexural rigidity (a measure of the bending resistance) of a plate or shell.
$D = E * t^3/(12 * (1 - \mu^2))$ **eq(31)**
β is the shell characteristic. <u>For a cylinder</u> $\beta = \{3 * (1 - \mu^2)/(R^2 * t^2)\}^{1/4}$ **eq(32)**
[**Note**: some references use λ for β.]
t is the shell thickness.
R is the cylinder mean radius.
E is the material property the modulus of elasticity (Young's modulus).
μ is the material property Poisson's ratio.
P is the loading intensity; the uniform internal pressure.

It is instructive to check what units the differential equation **(30)** has. The three groups of terms: d^4y/dx^4, $4*\beta^4*y$, and P/D all must have identical units. Check this out.

d^4y/dx^4 has units of Length/Length^4 = 1/Length^3 = Length^{-3}. This immediately sets the units of the other groups of terms.

In the group $4*\beta^4*y$ the 4 is a dimensionless constant and has no units. *y,* the radial deflection, has units of Length. In equation **(32)** Poisson's ratio μ is dimensionless and has no units, the group $(1- \mu^2)$ is also dimensionless as is the constant 3 hence the numerator is dimensionless, *R* and *t* in the denominator have units of Length. Therefore β, the shell characteristic <u>for a cylinder</u>, has units of 1/(Length^4)$^{1/4}$ = 1/Length = Length^{-1} and hence β^4*y has units of (Length^{-1})^4·Length = Length^{-4}·Length = Length^{-3}, as required.

[**Note**: Some references use the inverse of β, i.e. $1/\beta$ to give units of Length called the characteristic length of the cylinder. Not to be confused with the characteristic length used in fluid mechanics.]

In the group P/D, pressure *P* has units of Force/Length^2 = Force·Length^{-2}. In equation **(31)** Poisson's ratio μ is dimensionless and has no units, the group $(1- \mu^2)$ is also dimensionless as is the constant 12 hence the denominator is dimensionless. In the numerator *t* has units of Length and *E* the modulus of elasticity has units of stress/strain. Since strain is dimensionless; the units of *E* are stress i.e. Force/Length^2 = Force·Length^{-2}. Therefore the units of flexural rigidity *D* are: Force·Length^{-2}·Length^3 = Force·Length, and the units of P/D become: Force·Length^{-2}/(Force·Length) = Length^{-3}, as required.

The units of all the terms in equation **(30)** are in agreement.

In the theory of thin shell we are interested in getting the relationship of the end forces and moments acting on the shell alone. Setting the pressure to zero, $P = 0$, we have the homogeneous equation.

$$d^4y/dx^4 + 4*\beta^4*y = 0 \qquad \textbf{eq(33)}$$

[**Note**: this equation is analogous to well known equations in the solution of 'Beams on an Elastic Foundation' the classic text of which is by Hetényi, ref. 6.]

The general solution of equation **(33)** is:

$$y = e^{(\beta x)}*\{B_1\cos(\beta x) + B_2\sin(\beta x)\} + e^{(-\beta x)}*\{B_3\cos(\beta x) + B_4\sin(\beta x)\} \qquad \textbf{eq(34)}$$

Where: B_1, B_2, B_3, B_4 are four constants of integration found from the boundary conditions at the ends of the cylinder.

6.12.2 Cylindrical Shell, check the units.

To get useful results from equation **(34)** it is necessary to successively carry out differentiation to get the: slope, curvature, bending moment, and shear force. It is worth checking the units at each stage of the differentiation to establish that the LHS and RHS units are in agreement.

(1) Units of the constants. The first step is to establish what the units of the constants of integration, B_1, B_2, B_3, B_4 are? We know the units of the dependant variable, the radial deflection *y* on the LHS of **eq(34)** is Length so the units on the RHS must cancel to give units of length also. Check this out.

The units of β, the shell characteristic <u>for a cylinder</u> we have already established is: Length^{-1}. The units of *x*, the distance along the cylinder from the origin, is Length. Therefore, βx is dimensionless as are the trigonometric and exponential functions $\cos(\beta x)$, $\sin(\beta x)$ and $e^{(-\beta x)}$. This leaves only the constants of integration, B_1, B_2, B_3, B_4, which must therefore have the same units as the dependant variable, the radial deflection *y* on the LHS of **eq(34)**, i.e. B_1, B_2, B_3, and B_4 have units of Length.

(2) Units of the slope (rotation). For a <u>long</u> cylinder where the ends are remote from each other the equation **(34)** for the deflection can reduce to equation **(35)**:

$$y = e^{(-\beta x)}*\{B_3\cos(\beta x) + B_4\sin(\beta x)\} \qquad \textbf{eq(35)}$$

Differentiating equation **(35)** and rearranging we get the first derivative dy/dx which is the slope (i.e. the rotation) of the shell, theta θ in equation **(36)**. We can expect the rotation to be in radians, which are dimensionless. Check this out.

[**Note**: Strictly the slope dy/dx is tan θ. However, for small deformations the small angle approximation tan $\theta \approx \theta$ is acceptable.]

Recall that:	$y = k*\sin(\beta x)$ $dy/dx = k*\beta*\cos(\beta x)$, $y = k*\cos(\beta x)$ $dy/dx = -k*\beta*\sin(\beta x)$ $y = k*e^{-\beta x}$ $dy/dx = -k*\beta*e^{-\beta x}$

$$\theta = dy/dx = -B_3*\beta*e^{(-\beta x)}*\{\cos(\beta x) + \sin(\beta x)\} + B_4*\beta*e^{(-\beta x)}*\{\cos(\beta x) - \sin(\beta x)\} \quad \textbf{eq(36)}$$

The LHS differential has units of Length/Length i.e. dimensionless as expected for a rotation in radians. On the RHS βx is dimensionless as are the trigonometric and exponential functions $\cos(\beta x)$, $\sin(\beta x)$ and $e^{(-\beta x)}$. The constants of integration, B_3, B_4, have units of Length, we have just proved this above. β, the shell characteristic <u>for a cylinder</u> we have already established its units as: Length^{-1}. The units of $B_3*\beta$ and $B_4*\beta$ are $\text{Length}\cdot\text{Length}^{-1}$ therefore dimensionless. The units of the RHS of **eq(36)** do all cancel to be dimensionless as required for a rotation in radians.

(3) Units of the curvature. Differentiating equation **eq(36)** we get the second derivative d^2y/dx^2 which is the curvature of the shell, $1/r_x$ at any distance x from the origin, equation **(37)**.

$$1/r_x = d^2y/dx^2 = -2*B_3*\beta^2*e^{(-\beta x)}*\sin(\beta x) - 2*B_4*\beta^2*e^{(-\beta x)}*\cos(\beta x) \quad \textbf{eq(37)}$$

Where: r_x is the radius of curvature of the deformed shell at distance x from the origin, units of Length.

The LHS differential, the curvature, has units of: $\text{Length}/\text{Length}^2 = 1/\text{Length} = \text{Length}^{-1}$. On the RHS the 2's are dimensionless numbers and we already know the units of the other terms. The units of $B_3*\beta^2$ and $B_4*\beta^2$ are $\text{Length}\cdot(\text{Length}^{-1})^2 = \text{Length}\cdot\text{Length}^{-2} = \text{Length}/\text{Length}^2 = 1/\text{Length} = \text{Length}^{-1}$. The units of the RHS of **eq(37)** do cancel to give Length^{-1} as required for curvature.

(4) Units of the bending moment. The uniformly distributed bending moment *M1* around the circumference at any distance x from the origin is obtained by multiplying the curvature equation **(37)** by the flexural rigidity, D, equation **(31)**.

$$M1 = D*(d^2y/dx^2) = D/r_x \quad \textbf{eq(38)}$$

Therefore: $$M1 = -D*\{2*B_3*\beta^2*e^{(-\beta x)}*\sin(\beta x) + 2*B_4*\beta^2*e^{(-\beta x)}*\cos(\beta x)\} \quad \textbf{eq(39)}$$

Where: *M1* is the meridional bending moment per unit length of circumference, units: Force·Length/Length.

[**Note**: as mentioned before, it is common practice not to cancel out the lengths when checking these units so that it is clear that the numerator is a moment and the denominator is a unit length of the circumference.]

The RHS of equation **(39)** have to cancel to give units of: Force·Length/Length. Check this out.

On the RHS the 2's are dimensionless numbers and we already know the units of the other terms including the units of the flexural rigidity, D previously established as: Force·Length.
The units of $D*B_3*\beta^2$ and $D*B_4*\beta^2$ are: $\text{Force}\cdot\text{Length}\cdot\text{Length}\cdot(\text{Length}^{-1})^2 = \text{Force}\cdot\text{Length}^2\cdot\text{Length}^{-2} = \text{Force}\cdot\text{Length}/\text{Length}$. The units of the RHS of **eq(39)** do cancel to give Force·Length/Length as required for a moment per unit length of circumference.

(5) Units of the shear force. The uniformly distributed shear force V around the circumference at any distance x from the origin, can be obtained by differentiating the meridional bending moment **eq(39)**.

$$V = dM1/dx = D*(d^3y/dx^3) \quad \textbf{eq(40)}$$

Where: $$d^3y/dx^3 = 2*B_3*\beta^3*e^{(-\beta x)}*[\cos(\beta x) - \sin(\beta x)] + 2*B_4*\beta^3*e^{(-\beta x)}*[\cos(\beta x) + \sin(\beta x)] \quad \textbf{eq(41)}$$

Therefore: $$V = D*\{2*B_3*\beta^3*e^{(-\beta x)}*[\cos(\beta x) - \sin(\beta x)] + 2*B_4*\beta^3*e^{(-\beta x)}*[\cos(\beta x) + \sin(\beta x)]\} \quad \textbf{eq(42)}$$

Where: V is the shear force per unit length of circumference, units: Force/Length. The RHS of equation **(42)** have to cancel to give units of: Force/Length. Check this out.

On the RHS the 2's are dimensionless numbers and we already know the units of the other terms. The units of $D*B_3*\beta^3$ and $D*B_4*\beta^3$ are: Force·Length·Length·(Length^{-1})^3 = Force·Length^2·Length^{-3} = Force·Length^{-1} = Force/Length. The units of the RHS of **eq(42)** do cancel to give Force/Length as required for a shear force per unit length of circumference.

(6) Check the homogeneous equation. It is worth taking one additional step by 'differentiating to finality' (to coin a phrase). By differentiating **eq(41)** to get d^4y/dx^4 will allow us to check that the L.H.S. of the homogeneous equation, **eq(33)**, cancels out to zero.

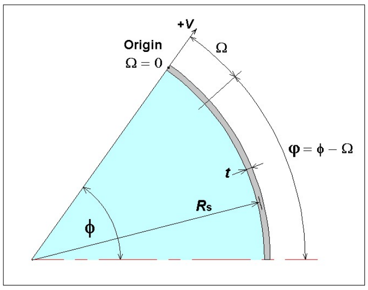
$$d^4y/dx^4 = -4*B_3*\beta^4*e^{(-\beta x)}*\cos(\beta x) - 4*B_4*\beta^4*e^{(-\beta x)}*\sin(\beta x) \qquad \textbf{eq(43)}$$

Substituting equation **(43)** for d^4y/dx^4 and equation **(35)** for y into equation **(33)** we see that the homogeneous equation does indeed cancel out to zero; a good indication that at least we may have got the differentiation correct.

The lesson here is: It is worth taking time to check the units of the differential equation at some point in the analysis. The units of the solution and subsequent derivatives should also be checked. A final check to show that the LHS of the homogeneous equation cancels out to zero gives further confidence that the differentiation has been carried out correctly.

6.12.3 Spherical Shell, check the units.

Before leaving this theory of shells it is worth remembering that shells come in several shapes e.g. cylindrical, spherical, conical, etc. Some care is required when dealing with different shapes. Let us consider the case of a spherical shell. There are several solutions for spherical shells but the most basic solution is Hetényi's Approximation I based on simplifications by Geckeler. The governing equation, for a very thin spherical shell, can be shown to be the following 4th order ordinary differential equation:



$$d^4V/d\varphi^4 + 4*\beta^4*V = 0 \qquad \textbf{eq(44)}$$

By comparison of **eq(44)** with **eq(33)** for cylindrical shells it is clear that both governing equations are of similar form.

Where: V the dependent variable is the Transverse Shear Force per unit circumference of the spherical shell with units of Force/Length.
φ the independent variable is the angle to a section across the shell thickness = (ϕ **-** Ω) with units of radians. [**Note**: for convenience, angles ϕ and Ω are often input in degrees. However, the mathematics requires a conversion to radians for φ.]

The general solution of equation **(44)** is:

$$V = e^{\beta\varphi}(B_1\cos(\beta\varphi) + B_2\sin(\beta\varphi)) + e^{-\beta\varphi}(B_3\cos(\beta\varphi) + B_4\sin(\beta\varphi)) \qquad \textbf{eq(45)}$$

By comparison of **eq(45)** with **eq(34)** for cylindrical shells it is clear that both solutions are of similar form. The four constants of integration, B_1, B_2, B_3, B_4 to be found from the edge boundary conditions will have units of the LHS of the equation **(45)** i.e. Force/Length. However, it is apparent that the shell characteristic β cannot be the same as that for cylindrical shells. With an angle φ in radians (i.e. dimensionless) then β must also be dimensionless in order that $\beta\varphi$ is dimensionless; as it must be to give a pure number for the index for Euler's number e and the trigonometric functions. Check this out.

For a spherical shell the shell characteristic is: $\beta = [3*(1-\mu^2)/(R_s/t)^2]^{1/4}$ **eq(46)**

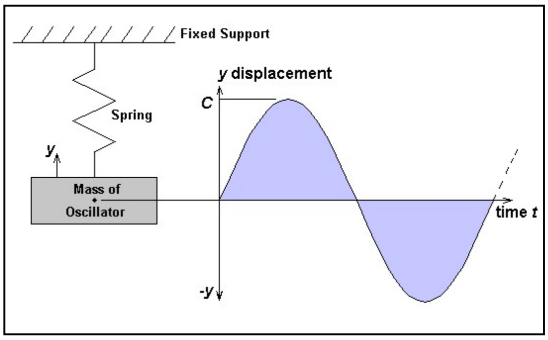
Where: R_s is the mean spherical radius with units of Length.
t is the shell thickness with units of Length.
μ is the material property Poisson's ratio and is dimensionless.

By comparison of equation **(46)** with equation **(32)** we can see that the shell characteristic for spheres and cylinders are different. The units of the numerator in **eq(46)** are dimensionless and since R_s/t is a ratio of Length/Length the denominator is also dimensionless, hence for a spherical shell the shell characteristic β is dimensionless as required.

The lesson here is: Do not assume that parameters with the same symbol are generally applicable. Using the shell characteristic equation for a cylinder in a spherical shell analysis (or vice versa) would lead to fatal results in the solution. Even where an equation is known to be generally applicable, such as the flexural rigidity *D*, which is applicable to both plates and shells, some references simplify the equation by ignoring Poisson's ratio so that the flexural rigidity is applied as $E*I$ as for a beam on an elastic foundation.

6.13 A Vibrating System Problem.

Dynamic systems involving vibrations are systems that have periodic motion. One common motion often assumed is that of a system in which a mass oscillates about an equilibrium position. The acceleration of the system is assumed to be directly proportional to the displacement from the equilibrium position and acts in the opposite direction to the displacement. i.e. acceleration = the negative of some constant times the displacement. This motion is often called Simple Harmonic Motion (SHM). Analysis of this motion involves detailed mathematical analysis and makes use of the idea of a point moving with uniform angular velocity ω around a circle that can be depicted by a sin curve of displacement *y* against time *t*.



Consider the motion of an oscillator such that the displacement $y = C*\sin(\omega*t + \phi)$ **eq(47)**

Where: *t* is time (the independent variable).

C is the amplitude of the oscillation.

ω is the angular velocity.

ϕ is the elapsed phase angle of the cycle (the units of which must be the same as for $\omega*t$).

The velocity is found by differentiating **eq(47)** with respect to time *t*:

dy/dt = velocity $V = C*\omega*\cos(\omega*t + \phi)$ **eq(48)**

The acceleration is found by differentiating the velocity, **eq(48)**, with respect to time:

d^2y/dt^2 = acceleration $a = -C*\omega^2*\sin(\omega*t + \phi) = -\omega^2*y$ **eq(49)**

This acceleration as a function of the displacement is the fundamental equation for SHM.

So much for the mathematics but what do the terms *C*, ω, and ϕ represent and what are their units?

6.13.1 Check the Units.

(1) Angular Velocity. ω is the angular velocity (also known as the rotational frequency or angular frequency or circular frequency or forcing frequency). It must have units of: Plane Angle/Time, but what angle is to be used? There are two common units for angles in use; degrees and radians. There are good mathematical reasons for using the radian. It simplifies the mathematics when dealing with trigonometric functions and is universally used in calculus. Therefore the units of angular velocity ω is radian/second.

[**Note**: Because the radian is the plane angle subtended at the centre of a circle by an arc length equal to the radius of the circle. Hence, radian angle is the ratio of: arc length/radius = radius/radius = 1 and is dimensionless as the units of Length/Length cancel out. In checking the units it is often common practice to simply write the units of ω as 1/s. However, it is important to know that the 1 in the numerator represents radian as it is perfectly possible to formulate the equations in terms of the 'ordinary' frequency in cycles/second also known as Hertz Hz, where Hz = 1/s and 1 represents cycles, as will be seen below. Just to confuse matters further, the SI unit for radian is rad. However, I am old enough to remember that at one time rad stood for absorbed radiation dose (now replaced by the SI derived unit the gray Gy). In this text I always spell out the units of plane angle as radian or radians so there is never any confusion.]

[**Note**: There are 2π radians in the circumference of a circle, therefore one radian = $360°/(2\pi)$ = about 57.2958° say 57.3 degrees. To convert radians to degrees multiply the radians by $180°/\pi$. To convert degrees to radians multiply the degrees by $\pi/180°$. Note also the trigonometric functions, sin *x*, cos *x* and tan *x* are dimensionless numbers and have no units.]

[**Note**: the units of $\omega*t$ = radian·s/s = radian·~~s~~/~~s~~ = radian and hence the units of the phase angle ϕ must also be in radian, otherwise there is no way that the group $(\omega*t + \phi)$ can be summed algebraically to give a sensible answer. If $\omega*t$ is converted to degrees then ϕ must also be in degrees.]

(2) Amplitude of the Oscillation. The units of the amplitude C have to be the same as the displacement x, i.e. a length, as the group $\sin(\omega*t + \phi)$ is dimensionless. The maximum value of $\sin(\omega*t + \phi)$ is = 1 hence the maximum value of the displacement $y = C$, and is known as the amplitude of the oscillation.

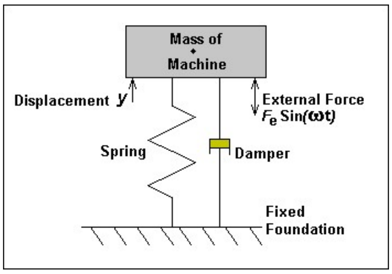
(3) Velocity. The units of velocity, $V = dy/dt$ can now be checked. The LHS of **eq(48)** requires units of length per unit time i.e. metres/second, m/s. Therefore the terms on the RHS of the equation must cancel to give units of m/s also. Check this out.

The group $\cos(\omega*t + \phi)$ is dimensionless. For displacement y in metres then the amplitude C is in metres also. We have already seen that ω is in radian/s. The units of the RHS of **eq(48)** are therefore: m·radian/s. Since radians are dimensionless the units of velocity V = m/s, as required.

(4) Acceleration. The units of acceleration, $a = d^2y/dt^2$ can now be checked. The LHS of **eq(49)** requires units of length per unit time squared i.e. metres/second squared, m/s^2. Therefore the terms on the RHS of the equation must cancel to give units of m/s^2 also. Check this out.

The group $\sin(\omega*t + \phi)$ is dimensionless. For displacement y in metres then the amplitude C is in metres also. We have already seen that ω is in radians/s. The units of the RHS of **eq(49)** are therefore: m.radians^2/s^2. Since radians are dimensionless the units of acceleration a = m/s^2, as required.

6.14 A Mass, Spring, Damper System subjected to a Forced Frequency.



The following Mass, Spring, Damper system is subjected to a periodic external force. We are required to find the force transmitted to the fixed foundations when the machine, of mass 2.5 tonnes, is acted upon by an external sinusoidal force of amplitude 3000 N at 15 Hz. The static deflection of the mass is 0.5 mm and the viscous damping coefficient is 180000 N/(m/s). Simple Harmonic Motion is to be assumed.

The equation of motion of the system is:

$$M_a*d^2y/dt^2 + c*dy/dt + k*y = F_e*\sin(\omega*t) \qquad \textbf{eq(50)}$$

Where: M_a is the mass of the machine.
c is the viscous damping coefficient.
k is the spring stiffness.
y is the displacement motion of the mass.
t is time.
F_e is the amplitude of the external sinusoidal force.
ω is the angular velocity of the forcing frequency in radians/s

The solution to equation **(50)** is well known and the force transmitted to the foundations can be calculated from the following two ratios:

ω/ω_o the ratio of the forcing frequency ω to the natural frequency ω_o of the undamped mass-spring system.

and

c/c_c the ratio of the actual damping c to the critical damping c_c of the system.

ω_o is the natural frequency of the undamped mass-spring system $\omega_o = \sqrt{(k/M_a)}$ **eq(51)**

The spring stiffness, k = weight of the mass/static deflection = F/δ **eq(52)**

Where: F is the weight of the machine = M_a*g **eq(53)**
M_a is the mass of the machine = 2500 kg
g is the acceleration due to gravity = 9.81 m/s^2
δ is the static deflection of the mass = 0.5 mm = 0.0005 m

c_c is the critical damping coefficient = $2*\omega_o*M_a$ **eq(54)**

6.14.1 Check the Units.

Check the units of eq(53); the weight of the machine should have units of force the newton, N. Units of M_a*g are kg·m/s^2, which is the base units of the newton a force as required.

Check the units of eq(52); the spring stiffness should have units of Force/Length, i.e. N/m. Units of F/δ are N/m, as required.

Check the units of eq(51); the natural frequency should have units of angular velocity, i.e. the same units as the forcing frequency ω radian/s. Check this out.

The spring stiffness, k has units N/m and the mass of the machine M_a has units of mass the kg. Therefore units of $\omega_o = \sqrt{(k/M_a)}$ = $\{N/(m\cdot kg)\}^{1/2}$. Since the base units for the newton is kg·m/s^2 we have $\{N/(m\cdot kg)\}^{1/2}$ = $\{kg\cdot m/(s^2\cdot m\cdot kg)\}^{1/2}$ which cancel $\{kg\cdot m/(s^2\cdot m\cdot kg)\}^{1/2}$ to give $\{1/s^2\}^{1/2}$ = 1/s. The 1 represents radians. The units of ω_o are radian/s, as required.

Check the units of the critical damping coefficient; this should have the same units as the viscous damping coefficient c. Check this out.

The viscous damping coefficient c has the units of: Force per unit Velocity, i.e N/(m/s) = N·s/m therefore the units of the critical damping coefficient c_c must be N·s/m. Now check **eq(54)**.

The 2 is a dimensionless number and has no units. The units of ω_o are radian/s as shown above. Therefore the units of c_c are: radian·kg/s. The radian is dimensionless and can be ignored from this units check. The units of mass in force and acceleration units is: N·s^2/m. Therefore the units of c_c are: N·s^2/(m·s) which cancels N·s^2/(m·s) to give N·s/m, as required.

All the units of equations **(51)** to **(54)** are correct.

Check the units of the differential equation, eq(50). As a final check it is worth doing a units check on the original differential equation, **eq(50)**, for consistency; each of the groups of terms must cancel to give identical units. Check this out.

M_a*d^2y/dt^2 has units of: kg·m/s^2 which is a force in newton N.

$c*dy/dt$ has units of: (N·s/m)·m/s which = N·s·m/(m·s) and cancels N·s·m/(m·s) to give units of force N.

$k*y$ has units of: (N/m)·m which = N·m/m cancel N·m/m to give units of force N.

$F_e*\sin(\omega*t)$ the amplitude of the external force F_e has units of force N. Sin($\omega*t$) is dimensionless and has no units. Therefore $F_e*\sin(\omega*t)$ has units of force N.

All the groups of terms in **eq(50)** have identical units as required.

6.14.2 Calculate the results.

The first step is to convert the forcing frequency ω from Hz i.e. cycles/second to radians/s.
$\omega = 2*\pi*$Frequency in Hz = $2*\pi*15$ = 94.2 radians/s

Now calculate the spring stiffness k.
Spring stiffness $k = F/\delta = M_a*g/\delta$ = 2500*9.81/0.0005 = 4.905×10^7 = 4.905E+7 N/m

The natural frequency ω_o of the system can now be calculated.
$\omega_o = \sqrt{(k/M_a)} = (k/M_a)^{1/2} = (4.905 \times 10^7/2500)^{1/2}$ = 140.1 radians/s

The ratio ω/ω_o = 94.2/140.1 = 0.67 and is dimensionless.

The forcing frequency ratio ω/ω_o = 0.67 is below $\sqrt{2}$, so there is potential for magnification of the external force to be transmitted to the foundations. It all depends on the damping in the system.

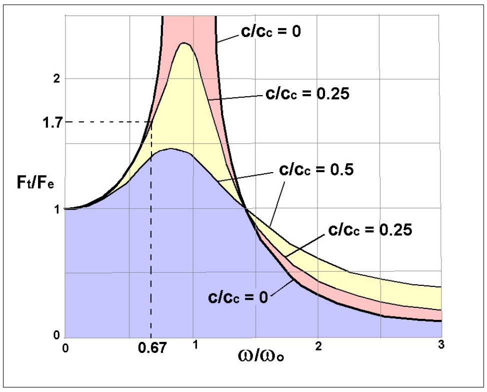
The viscous damping coefficient, c is given as = 180000 N·s/m

c_c is the critical damping coefficient = $2*\omega_o*M_a$

The critical damping coefficient c_c is now calculated.
$c_c = 2*\omega_o*M_a$ = 2*140.1*2500 = 700500 N·s/m

The ratio c/c_c = 180000/700500 = 0.25 and is dimensionless.

The magnification of the external force to be transmitted to the foundations can best be seen from the following diagram.



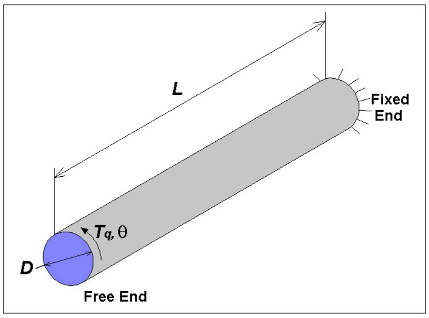
Entering the x-axis at ω/ω_o = 0.67 and plotting to the intersect for c/c_c = 0.25 gives a transmissibility ratio F_t/F_e = 1.7
i.e. the force transmitted to the foundations will be 1.7 times the applied external force.

Where: F_t is the force transmitted to the foundations
F_e is the amplitude of the external force = 3000 N

Therefore force transmitted to the foundations is:
$F_t = 1.7*F_e = 1.7*3000$ = 5100 N

6.15 A Torsion Problem.

This example introduces a torsion problem, which can be extended to show how the units of an integral can be checked. Consider the following solid round shaft of aluminium alloy 500 mm long subjected to a torque of 2000 Nm applied to a free end; the other end being fixed. It is required to size the shaft so it will not fail under the application of the torque and get the rotation of the free end. The aluminium alloy has a yield strength of 240 MPa and a modulus of elasticity of 69 GPa.



6.15.1 A Uniform Diameter Solid Shaft.

(1) Get a consistent set of units. The first thing to do is get these various parameters into a consistent set of units. Let us work with force in newton N and length in millimetres mm.

Length of the shaft is, L = 500 mm
Torque is, T_q = 2000 Nm = 2×10^6 Nmm
Yield strength is, S_y = 240 MPa = 240 N/mm^2
Modulus of elasticity is, E = 69 GPa = 69000 N/mm^2

(2) Get an allowable shear stress. The second thing that is required is an allowable stress so that the material will not yield when subjected to the applied torque. The torque on the shaft will result in a maximum shear stress τ_{max} on the outside of the shaft. Aluminium alloy is a ductile material, a suitable failure criterion is the maximum shear stress theory (also known as the Tresca yield criterion). In this theory it can be shown that yielding of the material will occur when the maximum shear stress is >= half the yield strength, i.e on the point of yielding $\tau_{max} = S_y/2$.

We will need a factor of safety (FOS), say 1.5.

Therefore the allowable shear stress will be limited to $\tau_{allow} = S_y/(2*\text{FOS}) = S_y/(2*1.5)$ = 240/3 = 80 N/mm^2.

(3) Calculate the required shaft diameter. The maximum shear stress of a solid shaft in torsion can be shown to be:

$$\tau_{max} = 16*T_q/(\pi*D^3) \qquad \textbf{eq(55)}$$

Rearranging **eq(55)** and putting $\tau_{max} = \tau_{allow}$ we have the required diameter of shaft.

$$D = \sqrt[3]{16*T_q/(\pi*\tau_{allow})} \qquad \textbf{eq(56)}$$

Now check the units of **eq(56)** to see that the units are Length in mm. The 16 and π are pure numbers and therefore dimensionless. Torque T_q has units of N·mm. The allowable stress τ_{allow} has units of

N/mm^2. Therefore the RHS of **eq(56)** has units of: $\{N{\cdot}mm/(N/mm^2)\}^{1/3} = \{N{\cdot}mm/(N{\cdot}mm^{-2})\}^{1/3} = \{N{\cdot}mm{\cdot}mm^2/N\}^{1/3} = \{N{\cdot}mm^3/N\}^{1/3} = \{mm^3\}^{1/3}$ = mm, as required.

The shaft diameter required is: $D = \sqrt[3]{\{16*2\times10^6/(\pi*80)\}}$ = 50.3 mm say use a shaft of 52 mm diameter.

(4) Calculate the rotation of the free end. The rotation of the free end due to the applied torque can be shown to be:

$$\theta = 32*T_q*L/(\pi*D^4*G) \qquad \textbf{eq(57)}$$

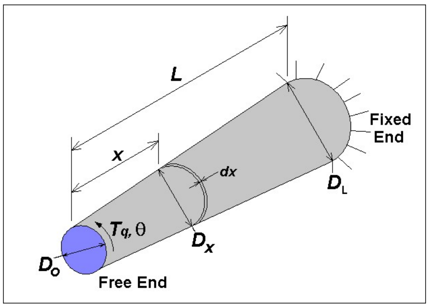
Where: θ is the rotation with units in radian.
G is the shear modulus sometimes called the modulus of rigidity. This can be calculated from the relationship: $G = E/(1 + \mu)$ **eq(58)**
E is the modulus of elasticity = 69000 N/mm^2 for aluminium alloy.
μ is the material property Poisson's ratio. A value of μ = 0.3 is applicable for metals.
Poisson's ratio is dimensionless as is the group $(1 + \mu)$. The shear modulus has the same units as the modulus of elasticity, i.e. stress. Therefore from **eq(58)**:
G = 69000/(1 + 0.3) = 53076.9 say 53000 N/mm^2.

Now check the units of **eq(57)** to see that they cancel out to be radians which are dimensionless. The 32 and π are pure numbers and therefore dimensionless. Torque T_q has units of N·mm. The length L and diameter D have units of mm and the shear modulus G has units of stress N/mm^2. Therefore the RHS of **eq(57)** has units of: $(N{\cdot}mm{\cdot}mm)/(mm^4{\cdot}N/mm^2) = N{\cdot}mm^2/(mm^4{\cdot}N{\cdot}mm^{-2}) = N{\cdot}mm^2/(mm^2{\cdot}N) = N{\cdot}mm^2/(mm^2{\cdot}N)$, which cancel out to be dimensionless as required.

The rotation of the free end of the uniform diameter shaft is:
$\theta = 32*2\times10^6*500/(\pi*52^4*53000) = 0.0263$ radians $= 180*0.0263/\pi = 1.506$ degrees.

6.15.2 A Linear Varying Diameter Solid Shaft.

Using the above example but with a shaft with a linear varying diameter, from 52 mm at the free end to say 62 mm at the fixed end, we can see how the units of an integral can be checked.



Replacing θ with $d\theta$, replacing L with dx and replacing D with D_x. Equation **(57)** becomes:

$$d\theta = 32*T_q*dx/(\pi*D_x^4*G) \qquad \textbf{eq(59)}$$

Where: $D_x = \{D_o + (D_L - D_o)*x/L\}$ the shaft diameter at any distance x from the free end. **eq(60)**
D_o is the shaft diameter at the free end (i.e. the diameter at x = 0) = 52 mm.
D_L is the shaft diameter at the fixed end (i.e. the diameter at $x = L$ = 500 mm) = 62 mm.
x is the distance along the shaft, to any section, from the free end.
L is the length of the shaft = 500 mm.

Differentiating equation **(60)**, with respect to distance x, gives:

$$dD_x/dx = (D_L - D_o)/L \qquad \textbf{eq(61)}$$

Therefore: $dx = L*dD_x/(D_L - D_o)$ **eq(62)**

Substitute equation **(62)** in **(59)** we have:

$$d\theta = \{32*T_q*L/[\pi*G*(D_L - D_o)]\}*dD_x/D_x^4 = k*dD_x/D_x^4 = k*(1/D_x^4)*dD_x \qquad \textbf{eq(63)}$$

Where: $k = \{32*T_q*L/[\pi*G*(D_L - D_o)]\}$ **eq(64)**

and the integral becomes:

$$\theta = k*\int_{x=0}^{x=L}(1/D_x^4)*dD_x \qquad \textbf{eq(65)}$$

We can now check the units of the integral **eq(65)**. The LHS is the rotation θ in radians therefore the RHS must cancel to be dimensionless. Check this out.

The integral sign $\int_{x=0}^{x=L}$ does not affect the units check.

The units are the product of the integrand $f(x) = 1/D_x^4$ = units of $1/mm^4$ and its differential $dx = dD_x$ = units of mm and the constant k in front of the integral sign **eq(64)**. The 32 and π are dimensionless. Torque T_q has units of N·mm. The length L, diameters D_x, D_L, D_o, and group $(D_L - D_o)$ all have units of mm and the shear modulus G has units of stress $N/mm^2 = N{\cdot}mm^{-2}$.

Therefore the constant k, equation **(64)**, has units of $\{N{\cdot}mm{\cdot}mm/[N{\cdot}mm^{-2}{\cdot}mm]\} = \{N{\cdot}mm^2/[N{\cdot}mm^{-1}]\} = \{\cancel{N}{\cdot}mm^3/\cancel{N}\} = mm^3$

Therefore the group $k{\cdot}(1/D_x^4){\cdot}dD_x$ the RHS of **eq(65)** = $k{\cdot}dD_x{\cdot}(1/D_x^4)$, ignoring the integral sign, has units of: $mm^3{\cdot}mm/mm^4 = \cancel{mm^4}/\cancel{mm^4}$, which cancel out to be dimensionless as required.

Integrating **eq(65)** we get **eq(66)**:

Recall that: $y = \int_{x=0}^{x=L} x^n dx = x^{n+1}/(n+1)$ and $y = \int_{x=0}^{x=L} (1/x^n)dx = [-1/(n-1)]*[1/x^{n-1}] = -1/[(n-1)*x^{n-1}]$	

$$\theta = k*[-1/(3*D_x^3)]_{x=0}^{x=L} \qquad \textbf{eq(66)}$$

Hence:

$$\theta = -(k/3)*[1/D_x^3]_{x=0}^{x=L} \qquad \textbf{eq(67)}$$

Therefore the result of the integration is:

$$\theta = -k*[1/D_L^3 - 1/D_o^3]/3 = -\{32*T_q*L/[\pi*G*(D_L - D_o)]\}*[1/D_L^3 - 1/D_o^3]/3 \qquad \textbf{eq(68)}$$

It is a good idea to again check the units to give some confidence that the integration has been carried out correctly. The constant k has units of mm^3 as before. The 3 in the denominator is a dimensionless number and has no units. The group $[1/D_L^3 - 1/D_o^3]$ has units of $1/mm^3$.

Therefore the RHS of **eq(68)** has units of: mm^3/mm^3, which cancel out $\cancel{mm^3}/\cancel{mm^3}$ to be dimensionless, i.e. radians, as required.

The result from equation **(68)**: at $x = L$ (the fixed end) $D_L = 62$ mm and at $x = 0$ (the free end) $D_o = 52$ mm becomes:

The rotation of the free end of the linearly varying diameter bar is.

$\theta = -\{32*2\times10^6*500/(3*\pi*53000*(62 - 52))\}*[1/62^3 - 1/52^3] = 0.0187$ radians
$= 180*0.0187/\pi = 1.07$ degrees.

The lesson here is: Like differentiation, it is worth taking time to check the units of the integration at each stage in the analysis. It will give some confidence that the analysis is proceeding correctly and will highlight at an early stage if something is going wrong.

7. Summary.

We can summarise what has been learned from the above discussion and examples:

1. Find out all you can about the equation and clarify the variables and their units.
2. Identify any constants (parameters, factors, coefficients, numbers, etc.) and establish if they have units or are they dimensionless pure numbers that can be ignored in the units check.
3. Note carefully any indices (exponents or powers), these should be dimensionless pure numbers, but they do need to be included in the units check.
4. Decide on a system of units to use, i.e. metric SI or Imperial.
5. Above all considerations get a consistent set of units; but take care to identify any equations that require to have input in particular units.
6. Become familiar with the units and the conversion between imperial and metric SI systems used in the particular industry, codes of practice and the equations commonly used in your day to day working.
7. Do not blindly put values into an equation without first considering if a units conversion is required.
8. Try to reduce the number of decimal multiples and hence the use of prefixes where possible.
9. If a set of calculations is being carried out. Check the units of each equation at each stage of the calculation. This is particularly important when carrying out differentiation or integration; check the units at each stage in the analysis. A final check to confirm that the homogeneous equation reduces to zero should also be done.
10. If an equation includes a series of groups of terms. Each group of terms must have identical units. The result cannot be added algebraically to give the correct answer unless each group of terms has identical units.
11. Check the units by cancellation. The units on both sides of the equation should be end up being identical; if the units are not identical then something is wrong.
12. Calculate the final result. Does it look reasonable? Build up experience in your industry or organisation so that you get a ‘feel’ for what is right or wrong.

8. References.

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Appendix 1: Rules of Indices.

The following are a few rules (laws) of indices (exponents or powers). The index (exponent or power) represents how many times to multiply the base quantity by, e.g. $x^3 = x*x*x$. The index is a pure (dimensionless) number. It can be positive or negative, integer or fraction, or complex.

Some Examples:

$x^0 = 1$ example: $5.67^0 = 1$ i.e. any number, except zero, with an index of zero is 1.

$x^1 = x$ example: $5.67^1 = 5.67$ i.e. any number with an index of 1 is the number itself.

$x^{-n} = 1/x^n$ example: $3^{-2} = 1/3^2 = 0.1111\ldots$
example: $3^{-1/2} = 1/3^{1/2} = 1/\sqrt{3} = 0.5773$

$\sqrt{x} = x^{1/2} = x^{0.5}$ example: $3^{1/2} = 1.732$

${}^n\sqrt{x} = x^{1/n}$ example: ${}^3\sqrt{3} = 3^{1/3} = 1.442$

$({}^m\sqrt{x})^n = x^{n/m}$ example: $({}^2\sqrt{3})^3 = (\sqrt{3})^3 = 3^{3/2} = 3^{1.5} = 5.196$

$x^{n/m} = (x^n)^{1/m}$ example: $(3^3)^{1/2} = (\sqrt{3})^3 = 3^{3/2} = 3^{1.5} = 5.196$

$x^{n/m} = (x^{1/m})^n$ example: $(3^{1/2})^3 = (\sqrt{3})^3 = 3^{3/2} = 3^{1.5} = 5.196$

$(x^n)^m = x^{n \cdot m}$ example: $(3^2)^3 = 3^{2*3} = 3^6 = 729$

Multiplication of terms of <u>same base</u>: $x^n \cdot x^m = x^{(n+m)}$
example: $3^5 * 3^2 = 3^{(5+2)} = 3^7 = 2187$
example: $3^2 * 3^{-5} = 3^{(2-5)} = 3^{-3} = 1/3^3 = 0.037$
example: $3^{-2} * 3^5 = 3^{(-2+5)} = 3^3 = 27$
example: $3^{-2} * 3^{-5} = 3^{(-2-5)} = 3^{-7} = 0.000457$

Division of terms of <u>same base</u>: $x^n/x^m = x^{(n-m)}$
example: $3^5/3^2 = 3^{(5-2)} = 3^3 = 27$
example: $3^2/3^5 = 3^{(2-5)} = 3^{-3} = 1/3^3 = 0.037$
example: $3^2/3^{-5} = 3^{(2+5)} = 3^7 = 2187$
example: $3^{-2}/3^5 = 3^{(-2-5)} = 3^{-7} = 1/3^7 = 0.000457$

Multiplication of terms of <u>different base</u>: $(x \cdot z)^n = x^n \cdot z^n$ example: $2^3 * 3^3 = 8*27 = 216$

Division of terms of <u>different base</u>: $(x/z)^n = x^n/z^n =$ example: $2^3/3^3 = 8/27 = 0.296$

Trigonometric Functions.
$\sin^n x$, $\cos^n x$, and $\tan^n x$.
Here $\sin^n x$ means $(\sin x)^n$, $\cos^n x$ means $(\cos x)^n$, $\tan^n x$ means $(\tan x)^n$. There is <u>one exception</u>; when n = -1. Then $\sin^{-1} x$ means the arcsin *x*, $\cos^{-1} x$ means the arccos *x*, and $\tan^{-1} x$ means the arctan *x*.
If you did not mean any of these arc functions then you need to use the brackets (or reciprocal), e.g. $(\sin x)^{-1} = 1/\sin x = \text{cosec } x$, or $(\cos x)^{-1} = 1/\cos x = \sec x$, or $(\tan x)^{-1} = 1/\tan x = \cot x$.

Index as a product of variables or constants.
If the index is a product of variables or constants then any units of the product must cancel to give a dimensionless number.
Example: $e^{(\beta x)}$ the units of the product βx must cancel to give a dimensionless number. Say the argument *x* has units of Length then the units of β must be Length^{-1} in order that the units of βx cancel. Alternatively if argument *x* is dimensionless then β must be dimensionless, e.g. if say the units of *x* were a plane angle in radians (which is a ratio and hence dimensionless) then β must be dimensionless so that βx cancel to give a dimensionless number.

Differentials such as: $d^n y/dt^n$
$d^n y$ means the n^{th} derivative of *y* hence the units of $d^n y$ are just the units of *y*, i.e. the d^n does NOT affect the units check.

dt^n means the derivative of *t* to the power *n*, i.e dt^n means $(dt)^n$ and the units of *dt* are the same as the units of *t* so the units of dt^n are the units of t^n. Therefore the units of $d^n y/dt^n$ is the units of y/t^n.

Example: suppose we have the derivative d^2y/dt^2 where the dependent variable y has units of Length, say metre m, and the independent variable (the argument) t has units of Time, say seconds s. Then the units of d^2y/dt^2 are = y/dt^2 = y/t^2 = m/s^2 i.e. units of acceleration.

Integrals such as: $k\int_{x=a}^{x=b} f(x)dx$

The integral sign $\int$ is dimensionless and does NOT affect the units check. The integrand $f(x)$, its differential dx and any constant(s) k in front of the integral do affect the units. The units are the product of: $k \cdot f(x) \cdot dx$.

Appendix 2: Notation.

A Pipe flow area when running full of fluid = $\pi * d^2/4$, units: Length^2. Refer Example 6.3.
A Surface area, units: Length^2. Refer Example 6.10.
a Acceleration, units: Length/Time^2. Refer Examples 6.1, 6.13.
a Slope (gradient) in straight line equation, units of Y- axis/units of X- axis.
a Crack size (crack depth), units: Length. Refer Example 6.7.
a_{crit} Critical crack size, units: Length. Refer Example 6.7.
(a) Indicates absolute pressure, e.g. bar(a) as opposed to bar(g) gauge pressure.
B_1, B_2, B_3, and B_4 Constants of integration. Refer Example 6.12.
b Y- axis intercept in straight line equation, units of Y- axis.
b Effective gasket seating width for bolted flanges, units mm or inches dependent upon the equation in use.
b_o Basic gasket seating width for bolted flanges, units mm or inches dependent upon the equation in use.
bar Pressure, 1 bar = 0.1 MN/m^2 = 0.1 MPa = 0.1 N/mm^2 = approximately 14.5 lbf/in^2.
C Amplitude of oscillation, units: Length. Refer Example 6.13.
C Chézy coefficient, units: Length$^{1/2}$/Time. Refer Example 6.4.
C Fracture mechanics, geometry correction factor, dimensionless. Refer Example 6.7.
C Thermal conductance, i.e. the heat transfer rate per unit temperature difference, units: Power/Thermodynamic Temperature Interval Units. Refer Example 6.9.
c Viscous damping coefficient, units: Force/Velocity = Force·Time/Length. Refer Example 6.14.
c_c Critical viscous damping coefficient, units: Force/Velocity = Force·Time/Length. Refer Example 6.14.
D Flexural rigidity of a plate or shell, $D = E * t^3/(12(1- \mu^2))$ with units: Force·Length. Refer Example 6.12.
D Diameter of a solid shaft, units: Length. Refer Example 6.15.
D_L Diameter at fixed end ($x = L$) of a solid shaft, units: Length. Refer Example 6.15.
D_o Diameter at free end (x = 0) of a solid shaft, units: Length. Refer Example 6.15.
D_x Solid shaft diameter at distance x from the free end, units: Length. Refer Example 6.15.
d pipe inside diameter, units: Length. Refer Examples 6.3, 6.4.
dD_x Differential of solid shaft diameter at distance x from free end, units: Length. Refer Example 6.15.
dT Temperature difference across the thickness of a cylinder, units: Thermodynamic Temperature Interval Units. Refer Examples 6.8, 6.9.
dx Differential of distance along a solid shaft from free end, units: Length. Refer Example 6.15.
$d\theta$ Differential of rotation, units: Plane Angle. Refer Example 6.15.
E Material property, the modulus of elasticity (Young's modulus), units: Force/Length^2. Refer Examples 6.7, 6.8, 6.11, 6.12, 6.15.
E_k Kinetic energy, units: Force·Length.
F Force, in SI units the newton N. Refer Examples 6.1, 6.2.
F Weight of a machine, in SI units the newton N. Refer Example 6.14.
F_e Amplitude of the external force, units: Force. Refer Example 6.14.
F_t Force transmitted to the foundations, units: Force. Refer Example 6.14.
f Friction factor (D'arcy coefficient), dimensionless. Refer Examples 6.3, 6.4.
f_1 Mohr circle, direct stress on face 1, units: Force/Length^2. Refer Example 6.6.
f_2 Mohr circle, direct stress on face 2, units: Force/Length^2. Refer Example 6.6.
G Gravitational constant ≈ 6.674×10^{-11} N·m^2/kg^2 Refer Example 6.2.
G Material property, shear modulus (modulus of rigidity), units: Force/Length^2. Refer Example 6.15.
G_c Material property, the toughness, sometimes called the 'critical strain energy release rate', units: Force/Length. Refer Example 6.7.
g Mass in grams.
g Acceleration due to gravity, units: Length/Time^2. Refer Examples 6.1, 6.3, 6.14.
(g) Indicates gauge pressure, e.g. bar(g) as opposed to bar(a) absolute pressure.
Hz Hertz, cycles/second, units = 1/s where the 1 stands for cycles.
h_f Loss of head due to fluid friction, units: Length. Refer Examples 6.3, 6.4.
h_i Heat transfer coefficient on the inside surface of a pipe, units: Power/(Length^2·Thermodynamic Temperature Interval Units). Refer Example 6.9.
h_o Heat transfer coefficient on the outside surface of a pipe, units: Power/(Length^2·Thermodynamic Temperature Interval Units). Refer Example 6.9.
J SI symbol of energy (work done), joule, = N·m = kg·m^2/s^2
K SI symbol for thermodynamic temperature, Kelvin.
K Crack stress intensity factor, units: Force/Length$^{3/2}$ or Stress·Length$^{1/2}$. Refer Example 6.7.
K_c Material property, the fracture toughness, sometimes called the 'critical stress intensity factor', units: Force/Length$^{3/2}$ or Stress·Length$^{1/2}$. Refer Example 6.7.
kg SI symbol for Mass, kilogram.

k a constant. Refer Example 6.15.
k Spring stiffness, units: Force/Length. Refer Example 6.14.
L Length of pipe or shaft, units: Length. Refer Examples 6.3, 6.4, 6.9, 6.15.
L SI symbol for litre, units: 10^{-3}m^3.
ln Logarithm to the base e.
M Mass of a body, units: Mass. Refer Example 6.1.
M_a Mass of a machine, units: Mass. Refer Example 6.14.
Mj1 Discontinuity bending moment per unit circumference at junction, component 1, units: Force·Length/Length. Refer Example 6.11.
Mj2 Discontinuity bending moment per unit circumference at junction, component 2, units: Force·Length/Length. Refer Example 6.11.
MPa SI symbol of pressure (stress), mega pascal, = MN/m^2 = N/mm^2
M1 Meridional bending moment per unit length of circumference, +ve if compression on the outside surface, units: Force·Length/Length. Refer Examples 6.11, 6.12.
M2 Circumferential (hoop) bending moment per unit length of circumference, +ve if compression on the outside surface, units: Force·Length/Length. Refer Example 6.11.
m SI symbol of Length, metre.
m_1 Mass of first body, units: Mass. Refer Example 6.2.
m_2 Mass of second body (mass of the Earth ≈ 5.972 × 10^{24} kg), units: Mass. Refer Example 6.2.
N SI symbol of force (weight), newton, = kg·m/s^2
N1 Meridional membrane force per length of circumference, units: Force/Length. Refer Example 6.11.
N2 Hoop membrane force per unit length of circumference, units: Force/Length. Refer Example 6.11.
P Power, SI units: watts W = J/s = N·m/s. Refer Example 6.3.
P Loading intensity, uniform internal pressure, units: Force/Length^2. Refer Example 6.12.
P Maximum safe pressure, units: Force/Length^2. Refer Example 6.5.
Pa SI symbol of pressure (stress), pascal, = N/m^2 = kg/(m·s^2)
P_{bar} Maximum safe pressure, units: bar. Refer Example 6.5.
Q Fluid flow rate, units: Length^3/Time. Refer Example 6.3.
q Heat transfer rate (heat flow rate), units: Power. Refer Examples 6.9, 6.10.
R Distance between the mass of two bodies, units: Length. Refer Example 6.2.
R Mean radius of a cylinder, units: Length. Refer Examples 6.5, 6.11, 6.12.
Re Reynolds Number = $\rho * v * d / \mu$ units: dimensionless. Refer Example 6.3.
R_s Mean radius of a sphere, units: Length. Refer Example 6.12.
rad SI symbol for plane angle, radian. [Note: I spell it out radian or radians in this text.]
r_i Pipe inside radius, units: Length. Refer Example 6.9.
r_o Pipe outside radius, units: Length. Refer Example 6.9.
r_x Radius of curvature, units: Length. $1/r_x$ is curvature, units: Length^{-1}. Refer Example 6.12.
S Allowable stress of a material, units: Force/Length^2. Refer Example 6.5.
S_y Material yield strength, units: Force/Length^2. Refer Example 6.15.
SBH Hoop bending stress on the surface, = $\pm 6*M2/T^2$, units: Force/Length^2. Refer Example 6.11.
SBL Meridional bending stress on the surface, = $\pm 6*M1/T^2$, units: Force/Length^2. Refer Example 6.11.
SH Hoop membrane stress, = $N2/T$, units: Force/Length^2. Refer Example 6.11.
SHi Hoop membrane + bending stress, inside surface, units: Force/Length^2. Refer Example 6.11.
SHo Hoop membrane + bending stress, outside surface, units: Force/Length^2. Refer Example 6.11.
SL Meridional membrane stress, = $N1/T$, units: Force/Length^2. Refer Example 6.11.
SLi Meridional membrane + bending stress, inside surface, units: Force/Length^2. Refer Example 6.11.
SLo Meridional membrane + bending stress, outside surface, units: Force/Length^2. Refer Example 6.11.
SV Average shear stress, = V/T, units: Force/Length^2. Refer Example 6.11.
s SI symbol for Time, second.
T Wall thickness of a cylinder, units: Length. Refer Example 6.11.
T_i Temperature at the inside surface of a pipe, units: Thermodynamic Temperature Interval Units, e.g. °C or K (or °F or °R). Refer Example 6.9.
T_o Temperature at the outside surface of a pipe, units: Thermodynamic Temperature Interval Units, e.g. °C or K (or °F or °R). Refer Example 6.9.
T_q Torque, units Force·Length. Refer Example 6.15.
T_1 Absolute temperature of a grey surface, units: kelvin K (or degree Rankine °R). Refer Example 6.10.
T_2 Absolute temperature of surroundings, units: kelvin K (or degree Rankine °R). Refer Example 6.10.
t Time, SI units: second s. Refer Examples 6.13, 6.14.
t Wall thickness of a cylinder or spherical shell, units: Length. Refer Examples 6.5, 6.12.
V Velocity of a body, units: Length/Time. Refer Example 6.13.
V Shear force per unit length of circumference, +ve outward. Refer Examples 6.11, 6.12.
Vj1 Discontinuity radial force per unit circumference at junction, component 1, units: Force/Length. Refer Example 6.11.

Vj2	Discontinuity radial force per unit circumference at junction, component 2, units: Force/Length. Refer Example 6.11.
V_o	Applied radial ring load, +ve inward, units: Force/unit Length of circumference. Refer Example 6.11.
v	Mean flow velocity of a fluid, units: Length/Time. Refer Examples 6.3, 6.4.
W	SI symbol of power, watt, = N·m/s = J/s = $kg \cdot m^2/s^3$
x	Independent variable (argument) in the RHS of an equation.
x	Distance from the origin to a section across a cylinder, units: Length. Refer Example 6.12.
x	Distance from the free end to a section along a solid shaft, units: Length. Refer Example 6.15.
Y	Radial deflection, units: Length. Refer Example 6.11.
y	Dependent variable, i.e. the LHS variable of an equation.
y	Radial deflection of a cylinder, units: Length. Refer Example 6.12.
y	Displacement of oscillator or mass, units: Length. Refer Examples 6.13, 6.14.
α	Coefficient of thermal expansion, units: 1/Thermodynamic Temperature Interval Units. e.g. 1/°C or 1/K (or 1/°F or 1/°R). Refer Example 6.8.
β	Shell characteristic. Refer Example 6.12. For a cylindrical shell, $\beta = \{3(1-\mu^2)/(R^2t^2)\}^{1/4}$ with units: Length^{-1}. For a spherical shell, $\beta = \{3(1-\mu^2)/(R_s/t)^2\}^{1/4}$ dimensionless.
δ	Static deflection of a mass, units: Length. Refer Example 6.14.
ε	Emissivity of a grey surface, units: dimensionless. Refer Example 6.10.
θ	Rotation or slope, units: Plane Angle. Refer Examples 6.12, 6.15.
λ	Thermal conductivity of a material, units: Power/(Length·Thermodynamic Temperature Interval Units). Refer Example 6.9.
μ	Dynamic viscosity of a fluid, units: Mass/(Length·Time). Refer Example 6.3.
μ	Material property, Poisson's ratio, dimensionless. Refer Examples 6.7, 6.8, 6.11, 6.12, 6.15.
ρ	Mass density, units: Mass/Length^3. Refer Example 6.3.
σ	Stress, units: Force/Length^2. Refer Example 6.7.
σ	Stefan-Boltzmann constant = 5.67×10^{-8} $W/(m^2 \cdot K^4)$. Refer Example 6.10.
σ_L	Longitudinal thermal bending stress, surface of a cylinder, units: Force/Length^2. Refer Example 6.8.
σ_H	Hoop thermal bending stress, surface of a cylinder, units: Force/Length^2. Refer Example 6.8.
σ_1	Mohr circle, first principal stress, units: Force/Length^2. Refer Example 6.6.
σ_2	Mohr circle, second principal stress, units: Force/Length^2. Refer Example 6.6.
τ	Mohr circle, shear stress on face 1 and 2, units: Force/Length^2. Refer Example 6.6.
τ_{allow}	Allowable shear stress, units: Force/Length^2. Refer Example 6.15.
τ_{max}	Maximum shear stress, units: Force/Length^2. Refer Example 6.15.
φ	Angle to a section across a spherical shell = $(\phi - \Omega)$, units: Plane Angle. Refer Example 6.12.
ϕ	Half arc angle of a spherical shell, units: Plane Angle. Refer Example 6.12.
ϕ	Elapsed phase angle of a cycle, units: Plane Angle. Refer Example 6.13.
Ω	Angle to a section across a spherical shell from the origin, units: Plane Angle. Refer Example 6.12.
ω	Angular velocity, units: Plane Angle/Time. Refer Example 6.13.
ω	Forcing frequency, Units: Plane Angle/Time. Refer Example 6.14.
ω_o	Natural frequency, units: Plane Angle/Time. Refer Example 6.14.

[**Note**: Plane Angle implies radians. Input in degrees is often more convenient than radians. However, a conversion from degrees to radians will be required before carrying out any mathematical analysis.]
